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Advances in Mathematics 229 (2012) 2525–2577
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Dynamics of McMullen maps

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Received 22 June 2010; accepted 19 December 2011
Available online 27 January 2012
Communicated by Kenneth Falconer

Abstract

In this article, we develop the Yoccoz puzzle technique to study a family of rational maps termed McMullen maps. We show that the boundary of the immediate basin of infinity is always a Jordan curve if it is connected. This gives a positive answer to the question of Devaney. Higher regularity of this boundary is obtained in almost all cases. We show that the boundary is a quasi-circle if it contains neither a parabolic point nor a recurrent critical point. For the whole Julia set, we show that the McMullen maps have locally connected Julia sets except in some special cases.
Published by Elsevier Inc.

MSC: 37F10; 37F45

Keywords: Local connectivity; Yoccoz puzzle; McMullen map

1. Introduction

The local connectivity of Julia sets for rational maps is a central problem in complex dynamical systems. It is well studied for classical types of rational maps, such as hyperbolic and semi-hyperbolic maps and geometrically finite maps [4,20,29]. The polynomial case is also well known [10,13,15,16,21,23]. For quadratic polynomials, Yoccoz proved that the Julia set is locally connected provided all periodic points are repelling and the map is not infinitely renormalizable [14,21]. Douady exhibited a striking example of an infinitely renormalizable quadratic

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0001-8708/$ – see front matter Published by Elsevier Inc.
doi:10.1016/j.aim.2011.12.026

polynomial with a non-locally connected Julia set [21]. For a general polynomial with connected Julia sets and without irrationally neutral cycles, Kiwi shows in [15] that the local connectivity of the Julia set is equivalent to the non-existence of wandering continua.

A powerful tool for studying the local connectivity of Julia sets for polynomials is the so-called 'Branner–Hubbard–Yoccoz puzzle' technique introduced by Branner and Hubbard [2]. This technique uses a natural method of construction involving finitely many periodic external rays together with an equipotential curve. However, for general rational maps, the situation is different, and the construction of the Yoccoz puzzle becomes quite involved, even impossible. Until now, the only known rational maps that admit Yoccoz puzzle structures were cubic Newton maps, whose Yoccoz puzzles were constructed by Roesch. In [26], Roesch applied Yoccoz puzzle techniques to show striking differences between rational maps and polynomials. The method also leads to the local connectivity of Julia sets except in some specific cases.

In this article, we present the Yoccoz puzzle structure for another family of rational maps known as McMullen maps. These maps are of the form

$$f_\lambda : z \mapsto z^n + \lambda/z^n, \quad \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\},\ n \geqslant 3.$$

The dynamics of this family of maps have been studied by Devaney and his group [5–8].

The Yoccoz puzzle differs for cubic Newton maps and McMullen maps in the following way. For cubic Newton maps, the Yoccoz puzzle is induced by a periodic Jordan curve that intersects the Julia set at countably many points. However, for McMullen maps, the element used to construct the Yoccoz puzzle is a periodic Jordan curve (this curve will be called the 'cut ray') that intersects the Julia set in a Cantor set. This type of Jordan curve is induced by some particular angle and can be viewed as an extension of the corresponding external ray (see Section 3.2).

We denote by B_λ the immediate basin of attraction of ∞. The topology of ∂B_λ is of special interest. Based on Yoccoz puzzle techniques and on combinatorial and topological analysis, we prove:

Theorem 1.1. *For any $n \geqslant 3$ and any complex parameter λ, if the Julia set $J(f_\lambda)$ is not a Cantor set, then ∂B_λ is a Jordan curve.*

This affirmatively answers a question posed by Devaney at the Snowbird Conference on the 25th Birthday of the Mandelbrot set [7]. For higher regularity of ∂B_λ, we show that ∂B_λ is a quasi-circle except in two special cases.

Theorem 1.2. *Suppose that the Julia set $J(f_\lambda)$ is not a Cantor set; then, ∂B_λ is a quasi-circle if it contains neither a parabolic point nor a recurrent critical point.*

Here, a recurrent critical point c on the Julia set of a rational map f is a critical point such that $c \in \omega(c)$, where $\omega(c)$ is the ω-limit set of c, defined as $\{z \in \overline{\mathbb{C}};\ \text{there exist } n_k \to \infty \text{ such that } z = \lim f^{n_k}(c)\}$. It follows from Proposition 7.5 that if ∂B_λ contains a parabolic point, then ∂B_λ is not a quasi-circle by the Leau–Fatou–Flower Theorem [21]. Whether ∂B_λ is a quasi-circle when ∂B_λ contains a recurrent critical point is still unknown.

For the topology of the Julia set, we show

Theorem 1.3. *Suppose f_λ has no Siegel disk and the Julia set $J(f_\lambda)$ is connected, then $J(f_\lambda)$ is locally connected in the following cases:*

1. *The critical orbit does not accumulate on the boundary* ∂B_λ.
2. f_λ *is neither renormalizable nor* $*$*-renormalizable.*
3. *The parameter* λ *is real and positive.*

See Section 5 for the definitions of renormalization and $*$-renormalization. Theorem 1.3 implies that the Julia set is locally connected except in some special cases. In fact, the theorem is stronger than the following statement:

Theorem 1.4. *Suppose* f_λ *has no Siegel disk and the Julia set* $J(f_\lambda)$ *is connected, then* $J(f_\lambda)$ *is locally connected if the critical orbit does not accumulate on the boundary* ∂B_λ.

Theorem 1.4 is an analogue of Roesch's Theorem [26]:

Theorem 1.5 *(Roesch). A genuine cubic Newton map without Siegel disks has a locally connected Julia set provided the orbit of the non-fixed critical point does not accumulate on the boundary of any invariant basin of attraction.*

We exclude the case $n = 2$ because it is impossible to find a non-degenerate critical annulus for the Yoccoz puzzle constructed in this paper. The existence of a non-degenerate critical annulus is technically necessary in our proof.

The paper is organized as follows:

In Section 2, we present some basic results on McMullen maps.

In Section 3, we construct 'cut rays', each of which is a type of Jordan curve that divides the Julia set into two different parts. We first construct a Cantor set of angles on the unit circle which is used to generate 'cut rays'. We then discuss the construction of 'cut rays' based on the work of Devaney [6].

In Section 4, basic knowledge of Yoccoz puzzles, graphs and tableaux are presented. The aim of this section is to find a Yoccoz puzzle with a non-degenerate critical annulus (see Section 4.2). A natural construction of the 'modified puzzle piece' is discussed (see Section 4.3).

In Section 5, we discuss the renormalizations of McMullen maps in the context of the puzzle piece.

In Section 6, we present a criterion of local connectivity. We introduce a '**BD** condition' on the boundary of the immediate basin of attraction. Such a condition can be considered as 'local semi-hyperbolicity'. We show that existence of the '**BD** condition' implies good topology.

In Section 7, we study the local connectivity of ∂B_λ in all possible cases and show that ∂B_λ enjoys higher regularity except in two special cases.

In Section 8, we study the local connectivity of the Julia set $J(f_\lambda)$ based on the 'Characterization of Local Connectivity' and the 'Shrinking Lemma'.

2. Preliminaries and notations

In this section, we present some basic results and notations for the family of rational maps

$$f_\lambda(z) = z^n + \lambda/z^n$$

where $\lambda \in \mathbb{C}^*$ and $n \geqslant 3$. This type of map is known as a ‘McMullen map’ because it was first studied by McMullen, who proved that when $|\lambda|$ is sufficiently small, and the Julia set is a Cantor set of circles [18].

For any $\lambda \in \mathbb{C}^*$, the map f_λ has a superattracting fixed point at ∞. The immediate basin of ∞ is denoted by B_λ, and the component of $f_\lambda^{-1}(B_\lambda)$ that contains 0 is denoted by T_λ. The set of all critical points of f_λ is $\{0, \infty\} \cup C_\lambda$, where $C_\lambda = \{\sqrt[2n]{\lambda}\omega;\ \omega^{2n} = 1\}$. Besides ∞, there are only two critical values for f_λ: $v_\lambda^+ = 2\sqrt{\lambda}$ and $v_\lambda^- = -2\sqrt{\lambda}$. In fact, there is only one critical orbit (up to a sign). Let $P(f_\lambda) = \overline{\bigcup_{n \geqslant 1} f_\lambda^k(C_\lambda)} \cup \{\infty\}$ be the post-critical set.

The Böttcher map ϕ_λ for f_λ is defined in a neighborhood of ∞ by $\phi_\lambda(z) = \lim_{k\to\infty}(f_\lambda^k(z))^{n^{-k}}$. The Böttcher map is unique if we require $\phi'_\lambda(\infty) = 1$. It is known that the Böttcher map ϕ_λ can be extended to a domain $\mathrm{Dom}(\phi_\lambda) \subset B_\lambda$ such that $\phi_\lambda : \mathrm{Dom}(\phi_\lambda) \to \{z \in \bar{\mathbb{C}}\colon |z| > R\}$ is a conformal isomorphism for some largest number $R \geqslant 1$. In particular, if B_λ contains no critical point other than ∞, then $\mathrm{Dom}(\phi_\lambda) = B_\lambda$; if B_λ contains a critical point $c \in \{0\} \cup C_\lambda$, then by ‘The Escape Trichotomy’ (Theorem 2.1), the Julia set $J(f_\lambda)$ is a Cantor set.

The Green function $G_\lambda : B_\lambda \to (0, \infty]$ is defined by

$$G_\lambda(z) = \lim_{k\to\infty} n^{-k} \log\bigl|f_\lambda^k(z)\bigr|.$$

By definition, $G_\lambda(f_\lambda(z)) = nG_\lambda(z)$ for $z \in B_\lambda$ and $G_\lambda(z) = \log|\phi_\lambda(z)|$ for $z \in \mathrm{Dom}(\phi_\lambda)$. The Green function G_λ can be extended to $A_\lambda = \bigcup_{k\geqslant 0} f_\lambda^{-k}(B_\lambda)$ by defining

$$G_\lambda(z) = n^{-k} G_\lambda\bigl(f_\lambda^k(z)\bigr) \quad \text{for } z \in f_\lambda^{-k}(B_\lambda).$$

In the following, for a set E in $\bar{\mathbb{C}}$ and $a \in \mathbb{C}$, let $aE = \{az;\ z \in E\}$, $a + E = \{a + z;\ z \in E\}$, $\bar{E}$ be the closure of E and $\mathrm{int}(E)$ be the interior of E.

Lemma 2.1 *(Symmetry of the dynamical plane). Let ω satisfy $\omega^{2n} = 1$; then,*

1. $\omega J(f_\lambda) = J(f_\lambda)$.
2. $G_\lambda(\omega z) = G_\lambda(z)$ *for* $z \in A_\lambda$.
3. $\omega \mathrm{Dom}(\phi_\lambda) = \mathrm{Dom}(\phi_\lambda)$, *and* $\phi_\lambda(\omega z) = \omega\phi_\lambda(z)$ *for* $z \in \mathrm{Dom}(\phi_\lambda)$.

Proof. For 1, because $A_\lambda = \{z \in \bar{\mathbb{C}};\ f_\lambda^k(z)$ tends to infinity as $k \to \infty\}$ and $f_\lambda^k(\omega z) = \pm f_\lambda^k(z)$ for $k \geqslant 1$, $f_\lambda^k(\omega z)$ tends toward infinity if and only if $f_\lambda^k(z)$ tends toward infinity as $k \to \infty$. Thus, $\omega A_\lambda = A_\lambda$. The conclusion follows from the fact that $J(f_\lambda) = \partial A_\lambda$.

2. By the definition of G_λ.

3. Because $\mathrm{Dom}(\phi_\lambda)$ is the connected component of $\{z \in B_\lambda;\ G_\lambda(z) > \log R\}$ that contains ∞, we conclude that $\omega\mathrm{Dom}(\phi_\lambda) = \mathrm{Dom}(\phi_\lambda)$. Note that $\phi_\lambda(\omega z)$ and $\omega\phi_\lambda(z)$ are two Riemann mappings of $\mathrm{Dom}(\phi_\lambda)$ onto $\{z \in \bar{\mathbb{C}};\ |z| > R\}$ with the same derivative at ∞, we have $\phi_\lambda(\omega z) = \omega\phi_\lambda(z)$ by the uniqueness of the Riemann mapping theorem. □

The non-escape locus of this family is defined by

$$M = \bigl\{\lambda \in \mathbb{C}^*;\ f_\lambda^k\bigl(v_\lambda^+\bigr) \text{ does not tend to infinity as } k \to \infty\bigr\}.$$

Lemma 2.2 *(Symmetry of the parameter plane). The non-escape locus M satisfies:*

1. *M is symmetric about the real axis.*
2. *$\nu M = M$ with $\nu^{n-1} = 1$.*
3. *For any line $\ell \in \{\epsilon\mathbb{R};\ \epsilon^{2n-2} = 1\}$, M is symmetric about ℓ.*

Proof. 1. Because $\overline{f_\lambda(\bar{z})} = f_{\bar{\lambda}}(z)$, the Critical orbit of f_λ and the critical orbit of $f_{\bar{\lambda}}$ are symmetric under the map $z \mapsto \bar{z}$, they either both remain bounded or both tend to infinity. Thus, M is symmetric about the real axis.

2. Let $\nu = e^{2\pi i/(n-1)}$ and $\varphi(z) = e^{\pi i/(n-1)}z$. For $k \geqslant 1$,

$$\varphi^{-1} \circ f_{\nu\lambda}^k \circ \varphi(z) = \begin{cases} (-1)^k f_\lambda^k(z), & n \text{ odd}, \\ f_\lambda^k(z), & n \text{ even}. \end{cases}$$

Thus, the critical orbit of f_λ tends toward infinity if and only if the critical orbit of $f_{\nu\lambda}$ tends toward infinity. Equivalently, $\lambda \in M$ if and only if $\nu\lambda \in M$.

3. The conclusion follows from 1 and 2. $\square$

From Lemma 2.2, f_λ and $f_{\lambda e^{2\pi i/(n-1)}}$ have the same dynamical properties and their Julia sets are identical up to a rotation. Thus, the fundamental domain of the parameter plane is $\{\lambda \in \mathbb{C}^*;\ \arg\lambda \in [0, \frac{2\pi}{n-1})\}$.

The following theorem of Devaney, Look and Uminsky gives a classification of Julia sets of different topological types [8].

Theorem 2.1 *(Devaney–Look–Uminsky). The Escape Trichotomy.*

1. *If $v_\lambda^+ \in B_\lambda$, then $J(f_\lambda)$ is a Cantor set.*
2. *If $v_\lambda^+ \in T_\lambda \neq B_\lambda$, then $J(f_\lambda)$ is a Cantor set of circles.*
3. *If $f_\lambda^k(v_\lambda^+) \in T_\lambda \neq B_\lambda$ for some $k \geqslant 1$, then $J(f_\lambda)$ is a Sierpiński curve, which is locally connected.*

In all other cases, the critical orbits remain bounded and the Julia set $J(f_\lambda)$ is connected.

For $n \geqslant 3$, it is known that the unbounded component of $\mathbb{C}^* - M$ consists of the parameters for which the Julia set is a Cantor set. This region is called a *Cantor set locus* (see Fig. 1). The component of $\mathbb{C}^* - M$ that contains a punctured neighborhood of 0 is the region in which the Julia set $J(f_\lambda)$ is a Cantor set of circles; this is referred to as the *McMullen domain* in honor of McMullen, who first discovered this type of Julia set. The complement of these two regions is the *connected locus*. The small copies of the quadratic Mandelbrot set correspond to the renormalizable parameters, while the 'holes' in the connected locus are always called *Sierpiński holes* according to Devaney. These regions correspond to the parameters for which the Julia set is a Sierpiński curve.

We will see later that, when the critical orbit tends to ∞, the boundary ∂B_λ is a quasi-circle if it is connected. Thus, this case is already well studied.

In this paper, we will restrict our attention to the parameters $\lambda \in \mathcal{H} = \{\lambda \in \mathbb{C}^*;\ \arg\lambda \in (0, \frac{2\pi}{n-1})\}$ for the most part because of the symmetry of the parameter plane. For these parameters,

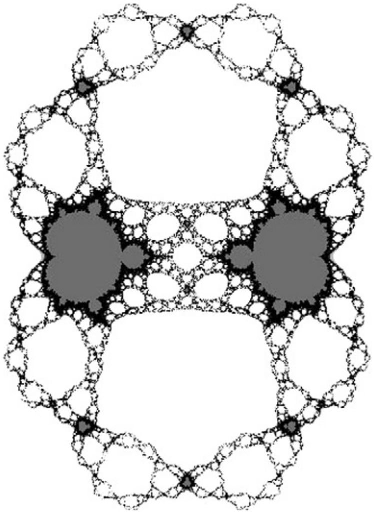

Fig. 1. Parameter plane for McMullen maps when $n = 3$.

we can develop Yoccoz puzzle techniques to study the local connectivity of Julia set. However, for real parameters, Yoccoz puzzle theory cannot be applied because of the absence of critical puzzle pieces. The real positive parameters will be considered separately in Section 7.3.

Therefore, if there is no further assumption, most discussions are based on the following:

Hypothesis. $\lambda \in \mathcal{H}$ and the critical orbits remain bounded, or equivalently, $C_\lambda \cap A_\lambda = \emptyset$.

2.1. Notations

Let $c_0 = c_0(\lambda) = \sqrt[2n]{\lambda}$ be the critical point that lies on $\mathbb{R}^+$ when $\lambda \in \mathbb{R}^+$ and varies analytically as λ ranges over $\mathcal{H}$. Let $c_k = c_0 e^{k\pi i/n}$ for $1 \leqslant k \leqslant 2n-1$. The critical points c_k with k even are mapped to $v_\lambda^+ = 2\sqrt{\lambda}$ while the critical points c_k with k odd are mapped to $v_\lambda^- = -2\sqrt{\lambda}$.

Let $\ell_k = c_k\mathbb{R}^+ (\mathbb{R}^+ := [0, +\infty])$ be the real straight line connecting the origin to ∞ and passing through c_k for $0 \leqslant k \leqslant 2n-1$. We call ℓ_k a *critical ray*. The closed sector bounded by ℓ_k and ℓ_{k+1} is denoted by S_k for $0 \leqslant k \leqslant n$. Define $S_{-k} = -S_k$ for $1 \leqslant k \leqslant n-1$. Therefore, the sectors are arranged counterclockwise about the origin as $S_0, S_1, \ldots, S_n, S_{-1}, \ldots, S_{-(n-1)}$ (see Fig. 2).

The critical value v_λ^+ always lies in S_0 because $\arg c_0 < \arg v_\lambda^+ < \arg c_1$ for all $\lambda \in \mathcal{H}$. Correspondingly, the critical value v_λ^- lies in S_n. It is easy to confirm that the image of ℓ_k under f_λ is a straight ray connecting one of the critical values to ∞; this ray is called a *critical value ray*. As a consequence, f_λ maps the interior of each of the sectors of $\{S_{\pm 1}, \ldots, S_{\pm(n-1)}\}$ univalently onto a region Υ_λ, which can be identified as the complex sphere $\bar{\mathbb{C}}$ minus two critical value rays.

Let $\mathcal{P}$ denote the set of all components of $\bigcup_{k\geqslant 0} f_\lambda^{-k}(B_\lambda)$. For $U \in \mathcal{P}$ and $v > 0$, let $\mathbf{e}(U, v) = \{z \in U;\ G_\lambda(z) = v\}$ be the equipotential curve. The annulus bounded by $\mathbf{e}(B_\lambda, v)$ and $\mathbf{e}(T_\lambda, v)$ is denoted by Q_v. We may choose a v large enough that ∂Q_v intersects with every critical ray at exactly two points (to see this, notice that the Böttcher map $\phi_\lambda : B_\lambda \to \bar{\mathbb{C}} - \bar{\mathbb{D}}$ acts like the identity map near ∞; thus, $\mathbf{e}(B_\lambda, v)$ looks like a circle when v is large. The curve $\mathbf{e}(T_\lambda, v)$ also

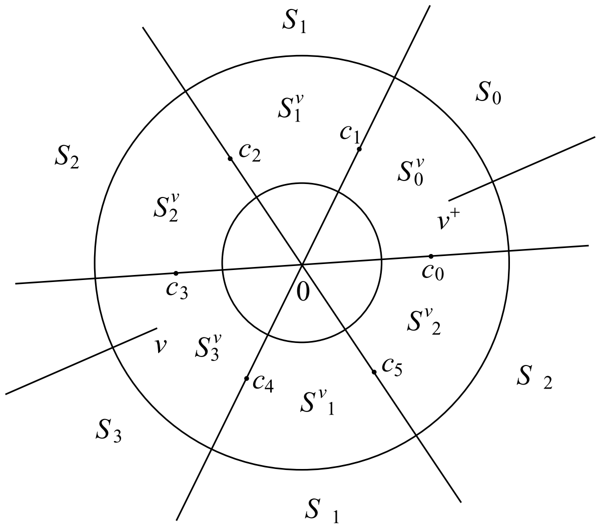

Fig. 2. Sectors in the dynamical plane when $n = 3$.

looks like a circle because $f_\lambda(\mathbf{e}(T_\lambda, v)) = \mathbf{e}(B_\lambda, nv)$ and f_λ acts like $z \mapsto \lambda/z^n$ near zero). The bounded and unbounded components of $\overline{\mathbb{C}} \setminus \mathbf{e}(B_\lambda, v)$ are denoted by $\mathbf{V}(v)$ and $\mathbf{U}(v)$, respectively.

Now, we define radial rays of U for every $U \in \mathcal{P} \setminus \{B_\lambda\}$. In the Hypothesis section, we see that there is a unique Riemann mapping $\phi_{T_\lambda} : T_\lambda \to \mathbb{D}$, such that

$$\phi_{T_\lambda}(z)^{-n} = \phi_\lambda\big(f_\lambda(z)\big), \quad z \in T_\lambda,\ \phi'_{T_\lambda}(0) = 1/\sqrt[n]{\lambda}.$$

The radial ray $R_{T_\lambda}(\theta)$ of angle θ is defined as $\phi_{T_\lambda}^{-1}((0,1)e^{2\pi i\theta})$. For $U \in \mathcal{P} \setminus \{B_\lambda, T_\lambda\}$, there is a smallest integer $k \geqslant 1$, such that $f_\lambda^k : U \to T_\lambda$ is a conformal map. The radial ray $R_U(\theta)$ is defined as the pullback of $R_{T_\lambda}(\theta)$ under f_λ^k.

Let $\mathbb{I} = \{0, n, \pm 1, \ldots, \pm(n-1)\}$ be an index set. $S_k^v = \overline{Q_v} \cap S_k$ for $k \in \mathbb{I}$ and $S^v = \bigcup_{k \in \mathbb{I}\setminus\{0,n\}} S_k^v$. The set of all points with orbits that remain in S^v under all iterations of f_λ is denoted by Λ_λ. Obviously, $\Lambda_\lambda = \bigcap_{k \geqslant 0} f_\lambda^{-k}(S^v)$.

For any $k \in \mathbb{I} \setminus \{0, n\}$, the map $f_\lambda : \mathrm{int}(S_k) \to \Upsilon_\lambda$ is a conformal map; its inverse is denoted by $h_k : \Upsilon_\lambda \to \mathrm{int}(S_k)$.

Given a point $z \in \Lambda_\lambda$, suppose $f_\lambda^k(z) \in S_{s_k}$ for $k \geqslant 0$ and define the itinerary of z as $\mathbf{s}_\lambda(z) = (s_0, s_1, s_2, \ldots)$. The itinerary is always well defined in the set Λ_λ because if some iteration $f_\lambda^k(z)$ lies on the boundary of two adjacent sectors, then the next iteration $f_\lambda^{k+1}(z)$ will lie inside $S_0 \cup S_n$.

Let $\Sigma = \{\mathbf{s} = (s_0, s_1, s_2, \ldots);\ s_k \in \mathbb{I} \setminus \{0, n\} \text{ for every } k \geqslant 0\}$ be the space of one-sided sequences of the symbols $\pm 1, \ldots, \pm(n-1)$. For $\mathbf{s} = (s_0, s_1, s_2, \ldots) \in \Sigma$,and the shift map $\sigma : \Sigma \to \Sigma$ is defined by $\sigma(\mathbf{s}) = (s_1, s_2, \ldots)$. If there is an integer $p > 0$ such that $s_{k+p} = s_k$ for all $k \geqslant 0$, we say the itinerary $\mathbf{s}$ is periodic and the least integer p is called the period of $\mathbf{s}$. In this case, $\mathbf{s}$ is also denoted by $(\overline{s_0, \ldots, s_{p-1}})$.

It is obvious that $\mathbf{s}_\lambda(f_\lambda(z)) = \sigma(\mathbf{s}_\lambda(z))$ for $z \in \Lambda_\lambda$.

Lemma 2.3. *The set Λ_λ is a Cantor set, and the itinerary map $\mathbf{s}_\lambda : \Lambda_\lambda \to \Sigma$ is bijective. Moreover, $\Lambda_\lambda \subset J(f_\lambda)$.*

Proof. First, note that for any $\lambda \in \mathcal{H}$, S^v is a compact subset of Υ_λ. With respect to the hyperbolic metric of Υ_λ and by the Schwarz Lemma, there is a number $\delta \in (0,1)$ such that for any $\mathbf{s} = (s_0, s_1, s_2, \ldots) \in \Sigma$ and any $m \geqslant 0$,

$$\text{Hyper.diam}\big(h_{s_0} \circ \cdots \circ h_{s_m}\big(S^v\big)\big) \leqslant \text{Hyper.diam}\big(S^v\big) \cdot \delta^m.$$

Thus, $\bigcap_{k \geqslant 0} h_{s_0} \circ \cdots \circ h_{s_k}(S^v)$ consists of a single point, say $z_{\mathbf{s}}$. Therefore, Λ_λ is a Cantor set, and the map $\mathbf{s}_\lambda : \Lambda_\lambda \to \Sigma$ defined by $\mathbf{s}_\lambda(z_{\mathbf{s}}) = \mathbf{s}$ is bijective.

When $\mathbf{s} = (\overline{s_0, \ldots, s_{m-1}}) \in \Sigma$ is a periodic itinerary of period m, then $z_{\mathbf{s}}$ is a fixed point of $h = h_{s_0} \circ \cdots \circ h_{s_{m-1}}$. Because $h : \text{int}(S^v) \to h(\text{int}(S^v)) \subset \text{int}(S^v_{s_0}) \Subset \text{int}(S^v)$ is strictly contractive, it follows by the Schwarz Lemma that the fixed point $z_{\mathbf{s}}$ is attracting. Therefore, $z_{\mathbf{s}}$ is a repelling periodic point of f_λ.

To show $\Lambda_\lambda \subset J(f_\lambda)$, it suffices to prove that any point of Λ_λ can be approximated by a sequence of repelling periodic points in Λ_λ. Suppose $z \in \Lambda_\lambda$. For any $\varepsilon > 0$, there is an integer $m > 0$ such that $\text{Hyper.diam}(S^v) \cdot \delta^m < \varepsilon$. Take a periodic itinerary $\mathbf{s} \in \Sigma$ with first m symbols that are the same as those of $\mathbf{s}_\lambda(z)$. (Notice that such an itinerary always exists.) Because the map $\mathbf{s}_\lambda$ is bijective, there is a unique point $w \in \Lambda_\lambda$ with $\mathbf{s}_\lambda(w) = \mathbf{s}$. The hyperbolic distance between z and w is smaller than ε. The previous argument implies that w is periodic and repelling. □

3. Cut rays in the dynamical plane

In this section, we will construct the 'cut ray', a type of Jordan curve that cuts the Julia set into two different parts. The construction is due to R. Devaney [6]. We give some additional properties that will be used in our paper.

We first construct a Cantor set of angles on the unit circle and use these angles to generate 'cut rays' as in [6]. These angles can be considered as a combinatorial invariant when the parameter λ ranges over $\mathcal{H}$.

To begin, we identify the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ with $(0,1]$. We say that three angles satisfy $t_1 \leqslant t_2 \leqslant t_3$ on $\mathbb{S}$ if t_1, t_2, t_3 are in counterclockwise order.

3.1. A Cantor set on the unit circle

In the following, we construct a subset Θ of (0,1]. The set Θ is a Cantor set and is used to generate 'cut rays' in the next section.

First, define a map $\tau : (0,1] \to (0,1]$ by $\tau(\theta) = n\theta \bmod 1$. Let $\Theta_k = (\frac{k}{2n}, \frac{k+1}{2n}]$ for $0 \leqslant k \leqslant n$ and $\Theta_{-k} = \Theta_k + \frac{1}{2}$ for $1 \leqslant k \leqslant n-1$. Obviously, $(0,1] = \bigcup_{k \in \mathbb{I}} \Theta_k$.

Define a map $\chi : \mathbb{I} \to \mathbb{N}$ by

$$\chi(k) = \begin{cases} k, & \text{if } 0 \leqslant k \leqslant n, \\ n-k, & \text{if } -(n-1) \leqslant k \leqslant -1. \end{cases}$$

For $k \in \mathbb{I}$, we have

$$\tau(\Theta_k) \supset \begin{cases} \bigcup_{j=1}^{n-1} \Theta_j, & \text{if } \chi(k) \text{ is even,} \\ \bigcup_{j=1}^{n-1} \Theta_{-j}, & \text{if } \chi(k) \text{ is odd.} \end{cases}$$

For $\theta \in (0, 1]$, suppose $\tau^k(\theta) \in \Theta_{s_k}$ for $k \geqslant 0$ and define the itinerary $\mathbf{s}(\theta)$ of θ by $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots)$.

Let Θ be the set of all angles $\theta \in (0, 1]$ with orbits that remain in $\mathcal{E} = \bigcup_{k=1}^{n-1}(\Theta_k \cup \Theta_{-k})$ under all iterations of τ. The set Θ can be written as $\Theta = \bigcap_{k\geqslant 0} \tau^{-k}(\mathcal{E}) = \bigcap_{k\geqslant 0} \tau^{-k}(\overline{\mathcal{E}})$. One can easily verify that Θ is a Cantor set.

The image of Θ under the itinerary map is denoted by $\Sigma_0 = \{\mathbf{s}(\theta);\ \theta \in \Theta\}$. One can easily verify that Σ_0 is a subspace of Σ that consists of all elements $\mathbf{s} = (s_0, s_1, s_2, \ldots) \in \Sigma$ such that for $k \geqslant 0$, if $\chi(s_k)$ is even, then $s_{k+1} \in \{1, \ldots, n-1\}$; if $\chi(s_k)$ is odd, then $s_{k+1} \in \{-1, \ldots, -(n-1)\}$.

The itinerary map $\mathbf{s} : \Theta \to \Sigma_0$ is bijective because for any $\mathbf{s} = (s_0, s_1, s_2, \ldots) \in \Sigma_0$, the intersection $\bigcap_{k\geqslant 0} \tau^{-k}(\Theta_{s_k})$ consists of a single point. In the following, we first construct an inverse map for $\mathbf{s}$ (Lemma 3.1).

Let $\mathbf{s} = (s_0, s_1, s_2, \ldots) \in \Sigma$. We define a map $\kappa : \Sigma \to (0, 1]$ by

$$\kappa(\mathbf{s}) = \frac{1}{2}\left(\frac{\chi(s_0)}{n} + \sum_{k\geqslant 1} \frac{|s_k|}{n^{k+1}}\right).$$

Lemma 3.1. $\kappa(\Sigma) = \Theta$ *and* $\kappa(\mathbf{s}(\theta)) = \theta$ *for all* $\theta \in \Theta$.

Proof. First, we show $\kappa(\mathbf{s}(\theta)) = \theta$ for $\theta \in \Theta$. Let $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots)$ and $\hat{\theta} = \kappa(\mathbf{s}(\theta))$. Because $\mathbf{s} : \Theta \to \Sigma_0$ is bijective, it suffices to show that $\mathbf{s}(\hat{\theta}) = \mathbf{s}(\theta)$.

It follows that $\hat{\theta} \in \Theta_{s_0}$ because

$$\frac{\chi(s_0)}{2n} < \hat{\theta} \leqslant \frac{1}{2}\left(\frac{\chi(s_0)}{n} + \sum_{k\geqslant 1} \frac{n-1}{n^{k+1}}\right) = \frac{\chi(s_0)}{2n} + \frac{1}{2n}.$$

For $k \geqslant 1$,

$$\tau^k(\hat{\theta}) = \begin{cases} \frac{1}{2}(\chi(s_0) + |s_1| + \cdots + |s_{k-1}|) + \frac{1}{2}\sum_{j\geqslant k} \frac{|s_j|}{n^{j-k+1}}, & \text{if } n \text{ is odd,} \\ \frac{|s_{k-1}|}{2} + \frac{1}{2}\sum_{j\geqslant k} \frac{|s_j|}{n^{j-k+1}}, & \text{if } n \text{ is even.} \end{cases}$$

Because $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots) \in \Sigma_0$, we have for $j \geqslant 1$,

$$\frac{|s_j|}{2} = \begin{cases} \frac{1}{2}(\chi(s_j) - \chi(s_{j-1})) \bmod 1, & \text{if } n \text{ is odd,} \\ \frac{1}{2}\chi(s_j) \bmod 1, & \text{if } n \text{ is even,} \end{cases}$$

and

$$\frac{\chi(s_{j-1})}{2} + \frac{|s_j|}{2n} = \frac{\chi(s_j)}{2n} \bmod 1.$$

Thus, we have

$$\tau^k(\hat{\theta}) = \frac{\chi(s_{k-1})}{2} + \frac{1}{2}\sum_{j\geqslant k}\frac{|s_j|}{n^{j-k+1}} = \frac{\chi(s_k)}{2n} + \frac{1}{2}\sum_{j\geqslant k+1}\frac{|s_j|}{n^{j-k+1}}.$$

This means $\tau^k(\hat{\theta}) \in \Theta_{s_k}$ for $k \geqslant 1$. Therefore, θ and $\hat{\theta}$ have the same itinerary.

In the following, we show $\kappa(\Sigma) = \Theta$. First, by the previous argument, $\Theta = \kappa(\Sigma_0) \subset \kappa(\Sigma)$. Conversely, for any $\mathbf{s} = (s_0, s_1, s_2, \ldots) \in \Sigma$, there is a unique sequence of symbols $\epsilon_1, \epsilon_2, \ldots \in \{\pm 1\}$, such that $\mathbf{s}^* = (s_0, \epsilon_1 s_1, \epsilon_2 s_2, \ldots) \in \Sigma_0$. Thus, $\kappa(\mathbf{s}) = \kappa(\mathbf{s}^*) \in \Theta$. □

Remark 3.1. For any $\mathbf{s} = (s_0, s_1, s_2, \ldots) \in \Sigma$, one can verify that

$$\kappa^{-1}\big(\kappa(\mathbf{s})\big) = \big\{(s_0, \pm s_1, \pm s_2, \ldots)\big\}.$$

Lemma 3.2. *The set Θ satisfies:*

1. $\tau(\Theta) = \Theta$.
2. $\Theta + \frac{1}{2} = \Theta$.
3. *Periodic angles are dense in Θ.*

Proof. 1. It is obvious that $\tau(\Theta) \subset \Theta$. τ is surjective because $\tau^{-1}(\theta) \cap \mathcal{E} \neq \emptyset$ for all $\theta \in \Theta$.

2. First note that $\mathcal{E} + \frac{1}{2} = \mathcal{E}$ mod 1. For $k \geqslant 1$, because $\tau^k(\theta + \frac{1}{2}) = \tau^k(\theta)$ when n is even and $\tau^k(\theta + \frac{1}{2}) = \tau^k(\theta) + \frac{1}{2}$ when n is odd, we have $\tau^k(\theta + \frac{1}{2}) \in \mathcal{E}$ if and only if $\tau^k(\theta) \in \mathcal{E}$. Thus, $\theta \in \Theta$ if and only if $\theta + \frac{1}{2} \in \Theta$.

3. Let $\theta \in \Theta$ with itinerary $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots)$. For any $k \geqslant 1$, either $(\overline{s_0, \ldots, s_k}) \in \Sigma_0$, or there is a symbol $s^*_{k+1} \in \{\pm 1, \ldots, \pm(n-1)\}$ such that $(\overline{s_0, \ldots, s_k, s^*_{k+1}}) \in \Sigma_0$. If $(\overline{s_0, \ldots, s_k}) \in \Sigma_0$, let $\theta_k = \kappa((\overline{s_0, \ldots, s_k}))$. Else, let $\theta_k = \kappa((\overline{s_0, \ldots, s_k, s^*_{k+1}}))$. It's obvious that θ_k is periodic. By Lemma 3.1, $\theta_k \in \Theta$ and

$$|\theta - \theta_k| \leqslant C(n) n^{-k} (\to 0 \text{ as } k \to \infty),$$

where $C(n)$ is a constant, depending only on n, which implies that periodic angles are dense in Θ. □

Remark 3.2. The Hausdorff dimension of Θ is $\frac{\log(n-1)}{\log n}$.

For $\lambda \in \mathcal{H}$ and $k \in \mathbb{I}$, let $\Theta^\lambda_k = \Theta_k + \frac{\arg c_0(\lambda)}{2\pi} = \Theta_k + \frac{\arg \lambda}{4n\pi}$ mod 1. Recall that for $\lambda \in \mathcal{H}$, $\arg \lambda \in (0, \frac{2\pi}{n-1})$. It is easy to check that

$$\tau\big(\Theta^\lambda_k\big) \supset \begin{cases} \bigcup_{j=1}^{n-1} \Theta^\lambda_j, & \text{if } \chi(k) \text{ is even}, \\ \bigcup_{j=1}^{n-1} \Theta^\lambda_{-j}, & \text{if } \chi(k) \text{ is odd}. \end{cases}$$

Again, we define Θ^λ as the set of all angles in $(0, 1]$ whose orbits remain in $\mathcal{E}^\lambda = \bigcup_{k=1}^{n-1}(\Theta^\lambda_k \cup \Theta^\lambda_{-k})$ under all iterations of τ. Thus, $\Theta^\lambda = \bigcap_{k\geqslant 0} \tau^{-k}(\mathcal{E}^\lambda)$. For $\theta \in (0, 1]$, suppose $\tau^k(\theta) \in \Theta^\lambda_{s_k}$ for $k \geqslant 0$ and define the itinerary of θ by $\mathbf{s}^\lambda(\theta) = (s_0, s_1, s_2, \ldots)$. It is easy to show that the itinerary map $\mathbf{s}^\lambda : \Theta^\lambda \to \Sigma_0$ is bijective.

Lemma 3.3. *$\Theta^\lambda = \Theta$ and for any $\theta \in \Theta$, $\mathbf{s}^\lambda(\theta) = \mathbf{s}(\theta)$.*

Proof. It suffices to show that if $\mathbf{s}^\lambda(\alpha) = \mathbf{s}(\beta)$ for $\alpha \in \Theta^\lambda$ and $\beta \in \Theta$, then $\alpha = \beta$.

First, note that $\Theta_k^\lambda \cap \Theta_k \neq \emptyset$ for any $k \in \mathbb{I}$. Suppose $\mathbf{s}^\lambda(\alpha) = \mathbf{s}(\beta) = (s_0, s_1, s_2, \ldots)$, and let $A_m = \bigcap_{0 \leqslant k \leqslant m} \tau^{-k}(\Theta_{s_k}^\lambda \cap \Theta_{s_k})$ for $m \geqslant 0$. By induction, we see that A_m is a connected interval of the form $(a_m, b_m]$ with $a_{m+1} > a_m$, $b_{m+1} < b_m$ and $n(b_{m+1} - a_{m+1}) = b_m - a_m$ for $m \geqslant 0$. Thus, $A_{m+1} \subset \overline{A_{m+1}} \subset A_m$ and $\bigcap_{k \geqslant 0} A_m = \bigcap_{k \geqslant 0} \overline{A_m}$ consists of a single point, say θ. On the other hand,

$$\{\theta\} = \bigcap_{k \geqslant 0} A_m = \left(\bigcap_{k \geqslant 0} \tau^{-k}\big(\Theta_{s_k}^\lambda\big)\right) \cap \left(\bigcap_{k \geqslant 0} \tau^{-k}(\Theta_{s_k})\right) = \{\alpha\} \cap \{\beta\}.$$

Thus, we have $\alpha = \beta = \theta$. $\square$

3.2. Cut rays

In this section, for any $\lambda \in \mathcal{H}$ and any $\theta \in \Theta$, we will construct a Jordan curve, say Ω_λ^θ, that cuts the dynamical plane of f_λ into two parts. The curve will meet the Julia set $J(f_\lambda)$ in a Cantor set of points. This kind of Jordan curve Ω_λ^θ will be called a 'cut ray' of angle θ. In the following, we construct such rays following a slightly different presentation from Devaney's in [6].

Recall that the itinerary map $\mathbf{s}_\lambda : \Lambda_\lambda \to \Sigma$ from a Cantor set onto a symbolic space is bijective. We first extend the definition of $\mathbf{s}_\lambda$ to a larger set. Let $E_\lambda = \bigcap_{k \geqslant 0} f_\lambda^{-k}(\bigcup_{j \in \mathbb{I} \setminus \{0,n\}} S_j)$ be the set of all points in the dynamical plane with orbits that remain in $\bigcup_{j \in \mathbb{I} \setminus \{0,n\}} S_j$ under all iterations of f_λ. By definition, E_λ is a compact subset of $\bar{\mathbb{C}}$ containing 0 and ∞. The assumption $\lambda \in \mathcal{H}$ implies that E_λ contains no critical points other than 0 and ∞.

Let $O_\lambda = \bigcup_{k \geqslant 0} f_\lambda^{-k}(\infty)$ be the grand orbit of ∞. The map $\mathbf{s}_\lambda : \Lambda_\lambda \to \Sigma$ can be extended to $\mathbf{s}_\lambda : E_\lambda \setminus O_\lambda \to \Sigma$ as follows: for any $z \in E_\lambda \setminus O_\lambda$, suppose $f_\lambda^k(z) \in S_{s_k}$ for $k \geqslant 0$; the itinerary of z is then defined by $\mathbf{s}_\lambda(z) = (s_0, s_1, s_2, \ldots)$. One can see that the map $\mathbf{s}_\lambda : E_\lambda \setminus O_\lambda \to \Sigma$ is well defined. (In fact, if $f_\lambda^n(z)$ lies on the intersection of two sectors, then $f_\lambda^{n+1}(z)$ will land on the critical value ray.)

Given an angle $\theta \in \Theta$ with itinerary $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots)$, it is easy to check that when n is odd, $\mathbf{s}(\theta + 1/2) = (-s_0, -s_1, -s_2, \ldots) = -\mathbf{s}(\theta)$ and that when n is even, $\mathbf{s}(\theta + 1/2) = (-s_0, s_1, s_2, \ldots)$. We consider the set of all points in $E_\lambda \setminus O_\lambda$ with itineraries that take the form $(s_0, \pm s_1, \pm s_2, \ldots)$. The closure of this set is denoted by ω_λ^θ:

$$\omega_\lambda^\theta = \overline{\big\{z \in E_\lambda \setminus O_\lambda;\ \mathbf{s}_\lambda(z) = (s_0, \pm s_1, \pm s_2, \ldots)\big\}} = \overline{\big\{z \in E_\lambda \setminus O_\lambda;\ \kappa\big(\mathbf{s}_\lambda(z)\big) = \theta\big\}}.$$

According to Devaney, the set ω_λ^θ is called a 'full ray' of angle θ. Let $\Omega_\lambda^\theta = \omega_\lambda^\theta \cup \omega_\lambda^{\theta+1/2}$; we call the set Ω_λ^θ a 'cut ray' of angle θ (or $\theta + 1/2$). One may verify that

$$\Omega_\lambda^\theta = \overline{\big\{z \in E_\lambda \setminus O_\lambda;\ \mathbf{s}_\lambda(z) = (\pm s_0, \pm s_1, \pm s_2, \ldots)\big\}} = \bigcap_{k \geqslant 0} f_\lambda^{-k}(S_{s_k} \cup S_{-s_k}).$$

We first give an intuitive description of the cut ray Ω_λ^θ. For $m \geqslant 0$, let

$$\Omega_{\lambda,m}^{\theta} = \bigcap_{0 \leqslant k \leqslant m} f_{\lambda}^{-k}(S_{s_k} \cup S_{-s_k}).$$

Note that the set $\Omega_{\lambda,0}^{\theta}$ is a union of the two closed sectors S_{s_0} and S_{-s_0}. $\Omega_{\lambda,1}^{\theta}$ is a string of four closed disks that lie inside $\Omega_{\lambda,0}^{\theta}$. Inductively, $\Omega_{\lambda,m}^{\theta}$ is a string of 2^{m+1} closed disks that are contained in $\Omega_{\lambda,m-1}^{\theta}$, and each of these disks meets exactly two others at the preimages of ∞. Hence, $\Omega_{\lambda,m}^{\theta}$ is a connected and compact set. One can show that $\Omega_{\lambda,m}^{\theta}$ converges to $\Omega_{\lambda}^{\theta} = \bigcap_{k \geqslant 0} \Omega_{\lambda,k}^{\theta}$ in Hausdorff topology as $m \to \infty$ (because a shrinking sequence of compact sets always converges in Hausdorff topology). Roughly, the set $\Omega_{\lambda,m}^{\theta}$ becomes thinner when m becomes larger and $\Omega_{\lambda,m}^{\theta}$ finally shrinks to $\Omega_{\lambda}^{\theta}$. It is therefore conjectured that $\Omega_{\lambda}^{\theta}$ is a Jordan curve. (A rigorous proof of this fact will be given in Proposition 3.3.)

By construction, the cut ray satisfies:

- $\Omega_{\lambda}^{\theta} = -\Omega_{\lambda}^{\theta}$.
- $\Omega_{\lambda}^{\theta} \setminus \{0, \infty\}$ is contained in the interior of $S_{s_0} \cup S_{-s_0}$.
- $f_{\lambda} : \Omega_{\lambda}^{\theta} \to \Omega_{\lambda}^{\tau(\theta)}$ is a two-to-one map.
- $\bigcup_{\theta \in \Theta} \Omega_{\lambda}^{\theta} = E_{\lambda}$.

Lemma 3.4. *Let $\lambda \in \mathcal{H}$; then, there is a constant $v > 0$ such that for any $\theta \in \Theta$,*

$$\overline{R_{\lambda}(\theta)} \cap \mathbf{U}(v) = \{z \in E_{\lambda} \cap \mathbf{U}(v);\ \mathbf{s}_{\lambda}(z) = \mathbf{s}(\theta)\}.$$

Proof. For any small number $\varepsilon > 0$, we define $\Theta_{k,\varepsilon}^{\lambda} = [\frac{\chi(k)}{2n} + \frac{\arg\lambda}{4n\pi} + \varepsilon, \frac{\chi(k)+1}{2n} + \frac{\arg\lambda}{4n\pi} - \varepsilon]$, $S_{k,\varepsilon} = \{z \in S_k \setminus \{0,\infty\};\ \arg z \in \Theta_{k,\varepsilon}^{\lambda}\} \cup \{0, \infty\}$ for $k \in \mathbb{I} \setminus \{0, n\}$. It is obvious that $S_{k,\varepsilon}$ is a closed subset of S_k. One can verify that there is an $\varepsilon > 0$ such that $\Theta^{\lambda} = \bigcap_{j \geqslant 0} \tau^{-j}(\bigcup_{k=1}^{n-1}(\Theta_{k,\varepsilon}^{\lambda} \cup \Theta_{-k,\varepsilon}^{\lambda}))$ and $E_{\lambda} = \bigcap_{k \geqslant 0} f_{\lambda}^{-k}(\bigcup_{j \in \mathbb{I}\setminus\{0,n\}} S_{j,\varepsilon})$. Thus, for any $\theta \in \Theta$ with $\mathbf{s}(\theta) = (s_0, s_1, \ldots)$, the cut ray $\Omega_{\lambda}^{\theta} = \bigcap_{k \geqslant 0} f_{\lambda}^{-k}(S_{s_k,\varepsilon} \cup S_{-s_k,\varepsilon})$. We fix such ε (notice that ε is independent of $\theta \in \Theta$).

Because $\phi_{\lambda}'(\infty) = 1$, we may choose $v = v(\varepsilon)$ large enough such that $|\arg z - \arg\phi_{\lambda}(z)| < \varepsilon$ for all $z \in \mathbf{U}(v)$. We define a map $\zeta : \mathbf{U}(v) \to \mathbb{S}$ by $\zeta(z) = \frac{\arg\phi_{\lambda}(z)}{2\pi}$. The map ζ satisfies $\zeta \circ f_{\lambda} = \tau \circ \zeta$.

If $z \in \overline{R_{\lambda}(\theta)} \cap \mathbf{U}(v)$ and $z \neq \infty$, then for any $k \geqslant 0$, $\arg\phi_{\lambda}(f_{\lambda}^k(z)) \in \Theta_{s_k,\varepsilon}^{\lambda}$. We conclude that $\arg f_{\lambda}^k(z) \in \Theta_{s_k}^{\lambda}$. Or, equivalently, $f_{\lambda}^k(z) \in S_{s_k}$ for all $k \geqslant 0$. Thus, $\mathbf{s}_{\lambda}(z) = \mathbf{s}(\theta)$.

On the other hand, for any $\infty \neq z \in E_{\lambda} \cap \mathbf{U}(v)$ with $\mathbf{s}_{\lambda}(z) = \mathbf{s}(\theta)$, we know from the above that $f_{\lambda}^k(z) \in S_{s_k,\varepsilon}$ for all $k \geqslant 0$, thus $\arg f_{\lambda}^k(z) \in \Theta_{s_k,\varepsilon}^{\lambda}$. It turns out that $\arg\phi_{\lambda}(f_{\lambda}^k(z)) = \tau^k(\zeta(z)) \in \Theta_{s_k}^{\lambda}$. By Lemma 3.3, $\mathbf{s}^{\lambda}(\zeta(z)) = \mathbf{s}(\theta) = \mathbf{s}^{\lambda}(\theta)$. Thus, we have $\zeta(z) = \theta$; this means $z \in R_{\lambda}(\theta) \cap \mathbf{U}(v)$. $\square$

Proposition 3.1. *For any $\lambda \in \mathcal{H}$ and any $\theta \in \Theta$, the external ray $R_{\lambda}(\theta)$ lands at a unique point $p_{\lambda}(\theta) \in \partial B_{\lambda}$ and $\overline{R_{\lambda}(\theta)} = \{z \in E_{\lambda} \setminus O_{\lambda};\ \mathbf{s}_{\lambda}(z) = \mathbf{s}(\theta)\} \cup \{\infty\} = \{z \in (E_{\lambda} \setminus O_{\lambda}) \cap B_{\lambda};\ \mathbf{s}_{\lambda}(z) = \mathbf{s}(\theta)\} \cup \{p_{\lambda}(\theta)\} \cup \{\infty\}$.*

Proof. Suppose $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots)$. Let $\ell_{\lambda}(v, \theta) = \{z \in R_{\lambda}(\theta);\ v \leqslant G_{\lambda}(z) \leqslant nv\}$ be the portion of $R_{\lambda}(\theta)$ that lies between two equipotential curves $\mathbf{e}(B_{\lambda}, v)$ and $\mathbf{e}(B_{\lambda}, nv)$. Based on Lemma 3.4, we can assume v large enough such that for any $\beta \in \Theta$, $\overline{R_{\lambda}(\beta)} \cap \mathbf{U}(v) =$

$\{z \in E_\lambda \cap \mathbf{U}(v);\ \mathbf{s}_\lambda(z) = \mathbf{s}(\beta)\}$. By pulling back $\ell_\lambda(v, \tau(\theta))$ by f_λ^{-1} to S_{s_0}, we can extend the portion of $\overline{R_\lambda(\theta)}$, say $\gamma_0 = \overline{R_\lambda(\theta)} \cap \mathbf{U}(v)$, to a longer one $\gamma_1 = h_{s_0}(\ell_\lambda(v, \tau(\theta))) \cup \gamma_0$. Obviously, $\gamma_1 \subset S_{s_0} \cap \overline{R_\lambda(\theta)}$. Continuing inductively, suppose we have already constructed a portion γ_k of $\overline{R_\lambda(\theta)}$; we then add a segment $h_{s_0} \circ \cdots \circ h_{s_k}(\ell_\lambda(v, \tau^{k+1}(\theta)))$ to γ_k and obtain $\gamma_{k+1} = \gamma_k \cup h_{s_0} \circ \cdots \circ h_{s_k}(\ell_\lambda(v, \tau^{k+1}(\theta)))$. By construction, one can confirm that $h_{s_0} \circ \cdots \circ h_{s_k}(\ell_\lambda(v, \tau^{k+1}(\theta))) \subset S_{s_0} \cap \overline{R_\lambda(\theta)}$, and that for any $z \in h_{s_0} \circ \cdots \circ h_{s_k}(\ell_\lambda(v, \tau^{k+1}(\theta)))$, $\mathbf{s}_\lambda(z) = (s_0, s_1, s_2, \ldots)$. It turns out that

$$R_\lambda(\theta) \setminus \gamma_0 = \bigcup_{k \geqslant 0} h_{s_0} \circ \cdots \circ h_{s_k}\big(\ell_\lambda\big(v, \tau^{k+1}(\theta)\big)\big).$$

In the following, we show that the external ray $R_\lambda(\theta)$ lands at ∂B_λ. Because $h_k : \Upsilon_\lambda \to \Upsilon_\lambda$ contracts the hyperbolic metric ρ_λ of Υ_λ for any $k \in \mathbb{I} \setminus \{0, n\}$, there is a constant $\delta \in (0, 1)$ such that

$$\rho_\lambda\big(h_k(x), h_k(y)\big) \leqslant \delta \rho_\lambda(x, y), \quad \forall x, y \in \overline{\mathbf{V}(nv)} \cap \left(\bigcup_{j \in \mathbb{I} \setminus \{0,n\}} S_j\right), \quad \forall k \in \mathbb{I} \setminus \{0, n\}.$$

Notice that $\bigcup_{\alpha \in \Theta} \ell_\lambda(v, \alpha) = E_\lambda \cap \{z \in B_\lambda;\ v \leqslant G_\lambda(z) \leqslant nv\}$ is a compact subset of Υ_λ, with respect to the hyperbolic metric of Υ_λ we have

$$\text{Hyper.length}\big(h_{s_0} \circ \cdots \circ h_{s_k}\big(\ell_\lambda\big(v, \tau^{k+1}(\theta)\big)\big)\big) = \mathcal{O}\big(\delta^k\big).$$

This implies that $R_\lambda(\theta) \setminus \gamma_0$ has finite hyperbolic length in Υ_λ; thus, the external ray $R_\lambda(\theta)$ lands at ∂B_λ. Let $p_\lambda(\theta)$ be the landing point. It is easy to confirm that $\mathbf{s}_\lambda(p_\lambda(\theta)) = \mathbf{s}(\theta)$ and $p_\lambda(\theta) \in \partial B_\lambda \cap \Lambda_\lambda$. Thus, we have

$$\begin{aligned}\overline{R_\lambda(\theta)} &\subset \big\{z \in (E_\lambda \setminus O_\lambda) \cap B_\lambda;\ \mathbf{s}_\lambda(z) = \mathbf{s}(\theta)\big\} \cup \big\{p_\lambda(\theta)\big\} \cup \{\infty\} \\ &\subset \big\{z \in E_\lambda \setminus O_\lambda;\ \mathbf{s}_\lambda(z) = \mathbf{s}(\theta)\big\} \cup \{\infty\}.\end{aligned}$$

Finally, we show $\overline{R_\lambda(\theta)} \supset \{z \in E_\lambda \setminus O_\lambda;\ \mathbf{s}_\lambda(z) = \mathbf{s}(\theta)\} \cup \{\infty\}$. For any $x \in \{z \in E_\lambda \setminus O_\lambda;\ \mathbf{s}_\lambda(z) = \mathbf{s}(\theta)\}$, we consider the orbit of x.

If the orbit of x remains bounded, then based on Lemma 2.3, we have $x \in \Lambda_\lambda$. Because $\mathbf{s}_\lambda|_{\Lambda_\lambda} : \Lambda_\lambda \to \Sigma$ is bijective and $\mathbf{s}_\lambda(x) = \mathbf{s}_\lambda(p_\lambda(\theta)) = \mathbf{s}(\theta)$, we conclude $x = p_\lambda(\theta) \in \overline{R_\lambda(\theta)}$.

If the orbit of x tends toward ∞, then by Lemma 3.4, there is an integer $M \geqslant 1$ such that $f_\lambda^M(x) \in R_\lambda(\tau^M(\theta))$. Note that for any $j \geqslant 0$, the above argument implies $R_\lambda(\tau^j(\theta)) \subset S_{s_j}$. Because $f_\lambda(R_\lambda(\tau^{k-1}(\theta))) = R_\lambda(\tau^k(\theta))$ and $h_{s_{k-1}}$ is the inverse branch of $f : \text{int}(S_{s_{k-1}}) \to \Upsilon_\lambda$, we conclude that for all $k \geqslant 1$, $h_{s_{k-1}}(R_\lambda(\tau^k(\theta))) = R_\lambda(\tau^{k-1}(\theta))$ and $h_{s_{k-1}}(f_\lambda^k(x)) = f_\lambda^{k-1}(x)$. It turns out that $x \in R_\lambda(\theta)$ and $\overline{R_\lambda(\theta)} \supset \{z \in E_\lambda \setminus O_\lambda;\ \mathbf{s}_\lambda(z) = \mathbf{s}(\theta)\} \cup \{\infty\}$. $\square$

Proposition 3.2. *For any $\lambda \in \mathcal{H}$ and any $\theta \in \Theta$ with itinerary $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots)$, the cut ray Ω_λ^θ satisfies:*

1. *Ω_λ^θ meets the Julia set $J(f_\lambda)$ in a Cantor set of points. More precisely, $\Omega_\lambda^\theta \cap J(f_\lambda) = (\kappa \circ \mathbf{s}_\lambda|_{\Lambda_\lambda})^{-1}(\{\theta, \theta + \frac{1}{2}\})$.*

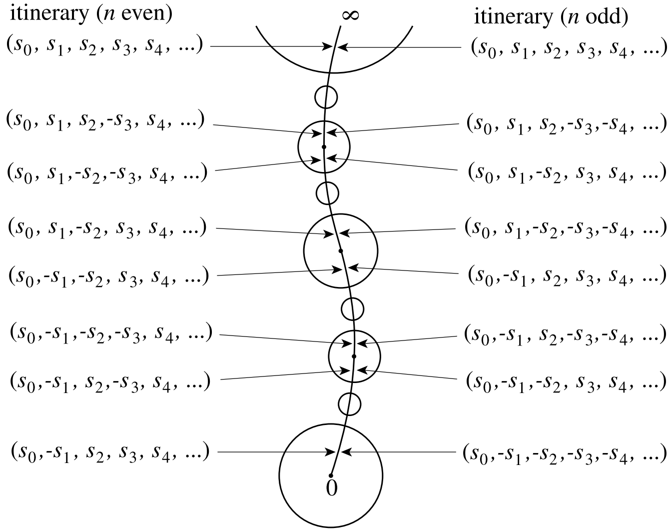

Fig. 3. Combinatorial structure of a full ray ω_λ^θ with $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots)$.

2. *Ω_λ^θ meets the Fatou set $F(f_\lambda)$ in a countable union of external rays and radial rays together with the preimages of ∞ that lie in the closure of these rays. More precisely,*

$$\Omega_\lambda^\theta \cap B_\lambda = R_\lambda(\theta) \cup R_\lambda\left(\theta + \frac{1}{2}\right) \cup \{\infty\},$$

$$\Omega_\lambda^\theta \cap T_\lambda = \begin{cases} h_{-s_0}(R_\lambda(\tau(\theta))) \cup h_{s_0}(R_\lambda(\tau(\theta) + \frac{1}{2})) \cup \{0\}, & \text{if } n \text{ is odd}, \\ h_{s_0}(R_\lambda(\tau(\theta) + \frac{1}{2})) \cup h_{-s_0}(R_\lambda(\tau(\theta) + \frac{1}{2})) \cup \{0\}, & \text{if } n \text{ is even}. \end{cases}$$

For any $U \in \mathcal{P} \setminus \{B_\lambda, T_\lambda\}$ with $U \cap \Omega_\lambda^\theta \neq \emptyset$, U is of the form $h_{b_0} \circ \cdots \circ h_{b_{k-1}}(T_\lambda)$, where $k \geqslant 1$ and $(b_0, \ldots, b_{k-1}) \in \{(\pm s_0, \ldots, \pm s_{k-1})\}$. Moreover,

$$\begin{aligned} &\Omega_\lambda^\theta \cap U \\ &\quad = h_{b_0} \circ \cdots \circ h_{b_{k-1}}\big(\Omega_\lambda^{\tau^k(\theta)} \cap T_\lambda\big) \\ &\quad = \begin{cases} h_{b_0} \circ \cdots \circ h_{b_{k-1}}(h_{-s_k}(R_\lambda(\tau^{k+1}(\theta))) \cup h_{s_k}(R_\lambda(\tau^{k+1}(\theta) + \frac{1}{2})) \cup \{0\}), & \text{if } n \text{ is odd}, \\ h_{b_0} \circ \cdots \circ h_{b_{k-1}}(h_{-s_k}(R_\lambda(\tau^{k+1}(\theta) + \frac{1}{2})) \cup h_{s_k}(R_\lambda(\tau^{k+1}(\theta) + \frac{1}{2})) \cup \{0\}), & \text{if } n \text{ is even}. \end{cases} \end{aligned}$$

See Fig. 3 for the combinatorial structure of a part of a cut ray.

Proof. 1. For $z \in \Omega_\lambda^\theta$, first note that $z \in \Omega_\lambda^\theta \cap J(f_\lambda)$ if and only if the orbit of z remains bounded, if and only if $z \in \Lambda_\lambda$ and $\mathbf{s}_\lambda(z) \in \{(\pm s_0, \pm s_1, \pm s_2, \ldots)\} = \kappa^{-1}(\{\theta, \theta + \frac{1}{2}\})$. Thus, we have $\Omega_\lambda^\theta \cap J(f_\lambda) = (\kappa \circ \mathbf{s}_\lambda|_{\Lambda_\lambda})^{-1}(\{\theta, \theta + \frac{1}{2}\})$.

2. Let U be a Fatou component such that $U \cap \Omega_\lambda^\theta \neq \emptyset$. Then, by 1, U is eventually mapped onto B_λ.

Case 1. $U = B_\lambda$. By Proposition 3.1, $\Omega_\lambda^\theta \cap B_\lambda \supset R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2}) \cup \{\infty\}$. On the other hand, for any $z \in (\Omega_\lambda^\theta \cap B_\lambda) \setminus \{\infty\}$, there is an integer $M \geqslant 1$ such that $f_\lambda^M(z) \in \mathbf{U}(v)$, where v is a positive constant chosen by Lemma 3.4. Because $\mathbf{s}_\lambda(f_\lambda^M(z)) \in \{(\pm s_M, \pm s_{M+1}, \pm s_{M+2}, \ldots)\}$, we conclude that the itinerary of $f_\lambda^M(z)$ must be the same as that of some angle $\beta \in \Theta$. Thus,

$$\mathbf{s}_\lambda\big(f_\lambda^M(z)\big) = \begin{cases} (s_M, s_{M+1}, s_{M+2}, \ldots) \text{ or } (-s_M, -s_{M+1}, -s_{M+2}, \ldots), & \text{if } n \text{ is odd,} \\ (s_M, s_{M+1}, s_{M+2}, \ldots), & \text{if } n \text{ is even.} \end{cases}$$

Case 1.1. n is odd. By Proposition 3.1, $f_\lambda^M(z) \in R_\lambda(\tau^M(\theta)) \cup R_\lambda(\tau^M(\theta) + \frac{1}{2})$. Note that $f_\lambda^{-1}(R_\lambda(\tau^M(\theta))) \cap (S_{s_{M-1}} \cup S_{-s_{M-1}}) \cap B_\lambda = R_\lambda(\tau^{M-1}(\theta))$, $f_\lambda^{-1}(R_\lambda(\tau^M(\theta) + \frac{1}{2})) \cap (S_{s_{M-1}} \cup S_{-s_{M-1}}) \cap B_\lambda = R_\lambda(\tau^{M-1}(\theta) + \frac{1}{2})$. We conclude that $f_\lambda^{M-1}(z) \in R_\lambda(\tau^{M-1}(\theta)) \cup R_\lambda(\tau^{M-1}(\theta) + \frac{1}{2})$. It turns out that $z \in R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2})$ by induction. So in this case, $\Omega_\lambda^\theta \cap B_\lambda = R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2}) \cup \{\infty\}$.

Case 1.2. n is even. By Proposition 3.1, $f_\lambda^M(z) \in R_\lambda(\tau^M(\theta))$. Because $f_\lambda^{-1}(R_\lambda(\tau^M(\theta))) \cap (S_{s_{M-1}} \cup S_{-s_{M-1}}) \cap B_\lambda = R_\lambda(\tau^{M-1}(\theta)) \cup R_\lambda(\tau^{M-1}(\theta) + \frac{1}{2})$, we have $f_\lambda^{M-1}(z) \in R_\lambda(\tau^{M-1}(\theta)) \cup R_\lambda(\tau^{M-1}(\theta) + \frac{1}{2})$. If $M = 1$, then $z \in R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2})$, and the proof is done. If $M > 1$, then we claim $f_\lambda^{M-1}(z) \in R_\lambda(\tau^{M-1}(\theta))$. This is because $f_\lambda^{-1}(R_\lambda(\tau^{M-1}(\theta) + \frac{1}{2})) \cap (S_{s_{M-2}} \cup S_{-s_{M-2}}) \cap B_\lambda = \emptyset$. Again, by induction, we have $z \in R_\lambda(\theta) \cup R_\lambda(\theta + \frac{1}{2})$ in this case.

Case 2. $U = T_\lambda$. In this case, if n is odd, then $f_\lambda(\Omega_\lambda^\theta \cap T_\lambda \cap S_{s_0}) = \Omega_\lambda^{\tau(\theta)} \cap B_\lambda \cap S_{-s_1} = R_\lambda(\tau(\theta) + \frac{1}{2}) \cup \{\infty\}$ and $f_\lambda(\Omega_\lambda^\theta \cap T_\lambda \cap S_{-s_0}) = \Omega_\lambda^{\tau(\theta)} \cap B_\lambda \cap S_{s_1} = R_\lambda(\tau(\theta)) \cup \{\infty\}$. So $\Omega_\lambda^\theta \cap T_\lambda = h_{-s_0}(R_\lambda(\tau(\theta))) \cup h_{s_0}(R_\lambda(\tau(\theta) + \frac{1}{2})) \cup \{0\}$; if n is even, then $f_\lambda(\Omega_\lambda^\theta \cap T_\lambda \cap S_{s_0}) = f_\lambda(\Omega_\lambda^\theta \cap T_\lambda \cap S_{-s_0}) = \Omega_\lambda^{\tau(\theta)} \cap B_\lambda \cap S_{-s_1} = R_\lambda(\tau(\theta) + \frac{1}{2}) \cup \{\infty\}$. So $\Omega_\lambda^\theta \cap T_\lambda = h_{s_0}(R_\lambda(\tau(\theta)) + \frac{1}{2}) \cup h_{-s_0}(R_\lambda(\tau(\theta) + \frac{1}{2})) \cup \{0\}$.

Case 3. $U \in \mathcal{P} \setminus \{B_\lambda, T_\lambda\}$. In this case, there is a smallest integer $k \geqslant 1$ such that $f_\lambda^k(U) = T_\lambda$. Because $f_\lambda^k : U \to T_\lambda$ is a conformal map and for any $0 \leqslant j \leqslant k-1$, $f_\lambda^j(U)$ lies inside some sector S_{k_j}, we conclude U must take the form $h_{b_0} \circ \cdots \circ h_{b_{k-1}}(T_\lambda)$ for some $(b_0, \ldots, b_{k-1}) \in \{(\pm s_0, \ldots, \pm s_{k-1})\}$. By pulling back $f_\lambda^k(U \cap \Omega_\lambda^\theta) = \Omega_\lambda^{\tau^k(\theta)} \cap T_\lambda$ via f_λ^k, we have $\Omega_\lambda^\theta \cap U = h_{b_0} \circ \cdots \circ h_{b_{k-1}}(\Omega_\lambda^{\tau^k(\theta)} \cap T_\lambda)$. The conclusion follows by Case 2. □

Proposition 3.3. *For any $\lambda \in \mathcal{H}$ and any $\theta \in \Theta$, the cut ray Ω_λ^θ is a Jordan curve* (*see Fig.* 4).

Proof. Suppose $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots)$. For $k \geqslant 0$, define

$$\hat{\Omega}_{\lambda,0}^{\tau^k(\theta)} = \Omega_\lambda^{\tau^k(\theta)} \cup S_{s_k}^v \cup S_{-s_k}^v, \qquad \hat{\Omega}_{\lambda,k}^\theta = \bigcap_{0 \leqslant j \leqslant k} f_\lambda^{-j}\big(\hat{\Omega}_{\lambda,0}^{\tau^j(\theta)}\big).$$

The set $\hat{\Omega}_{\lambda,k}^\theta$ is connected and compact, and it contains Ω_λ^θ. It is easy to check that $\hat{\Omega}_{\lambda,k}^\theta \supset \hat{\Omega}_{\lambda,k+1}^\theta$ and $\bigcap_{k \geqslant 0} \hat{\Omega}_{\lambda,k}^\theta = \Omega_\lambda^\theta$. In the following discussion, we can assume k is sufficiently large that $\hat{\Omega}_{\lambda,k}^\theta$ avoids the critical values $v_\lambda^\pm$. Let D_k^+ be the component of $\overline{\mathbb{C}} \setminus \hat{\Omega}_{\lambda,k}^\theta$ that contains v_λ^+ and

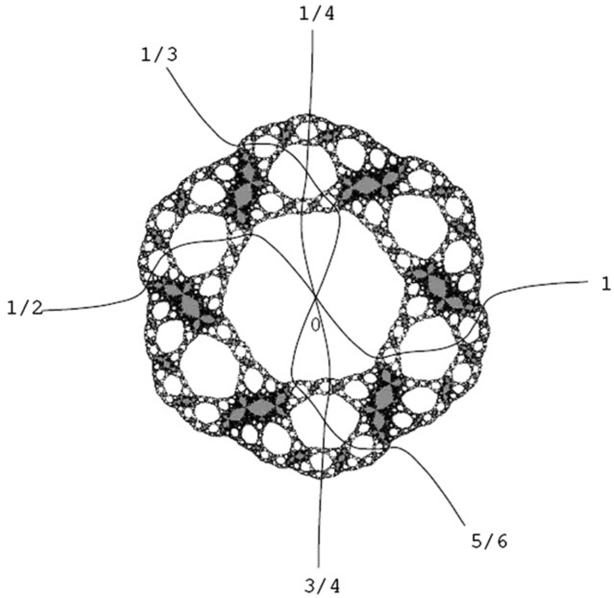

Fig. 4. Cut rays with angles $1/4$, $1/3$, $1/2$ when $n = 3$.

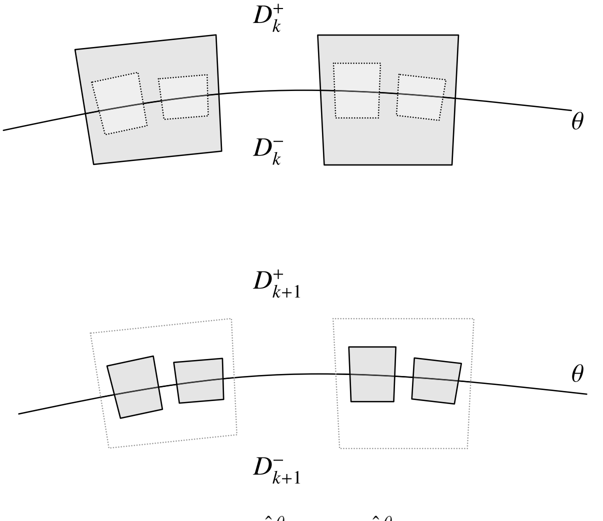

Fig. 5. The cut ray union with the shadow regions is $\hat{\Omega}_{\lambda,k}^{\theta}$ (resp. $\hat{\Omega}_{\lambda,k+1}^{\theta}$). The two components of $\bar{\mathbb{C}} - \hat{\Omega}_{\lambda,k}^{\theta}$ (resp. $\bar{\mathbb{C}} - \hat{\Omega}_{\lambda,k+1}^{\theta}$) are D_k^+ and D_k^- (resp. D_{k+1}^+ and D_{k+1}^-).

D_k^- be the component of $\overline{\mathbb{C}} \setminus \hat{\Omega}_{\lambda,k}^{\theta}$ that contains v_λ^-. Let $D_\infty^+ = \bigcup_{k \geqslant 0} D_k^+$ and $D_\infty^- = \bigcup_{k \geqslant 0} D_k^-$; then, $D_\infty^+ \cup D_\infty^- \cup \Omega_\lambda^\theta = \overline{\mathbb{C}}$ (Fig. 5).

We first construct a Cantor set on $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. Let $E_1 = (5/24, 13/24)$, $E_2 = (17/24, 25/24)$ be two open intervals on $\mathbb{S}$ and ζ be the map $t \mapsto 3t \bmod \mathbb{Z}$. By definition, $\zeta(E_i) \supset \overline{E_1 \cup E_2}$. Let $T_k = \bigcap_{0 \leqslant j \leqslant k} \zeta^{-j}(E_1 \cup E_2)$. Then, $T_k \supset T_{k+1}$ and T_k has 2^{k+1} components. The intersection

$\bigcap_{k\geqslant 0} T_k$ is denoted by T_∞. Because $T_\infty = \bigcap_{k\geqslant 0}\zeta^{-k}(E_1 \cup E_2) = \bigcap_{k\geqslant 0}\zeta^{-k}(\overline{E_1} \cup \overline{E_2})$, we conclude that T_∞ is a Cantor set.

Now, we define two sequences of Jordan curves $\{\gamma_k^+ : \mathbb{S} \to \partial D_k^+\}$, $\{\gamma_k^- : \mathbb{S} \to \partial D_k^-\}$ in the following manner: for k large enough,

1. $\gamma_{k+1}^+|_{\mathbb{S}\setminus T_k} = \gamma_k^+|_{\mathbb{S}\setminus T_k} = \gamma_k^-|_{\mathbb{S}\setminus T_k} = \gamma_{k+1}^-|_{\mathbb{S}\setminus T_k}$.
2. $\gamma_k^+(\mathbb{S}\setminus T_k) = \Omega_\lambda^\theta \cap \partial D_k^+ = \Omega_\lambda^\theta \cap \partial D_k^- = \gamma_k^-(\mathbb{S}\setminus T_k)$.
3. $\gamma_k^+(T_k) = \partial D_k^+ \setminus \Omega_\lambda^\theta$, $\gamma_k^-(T_k) = \partial D_k^- \setminus \Omega_\lambda^\theta$.

In the following, we show that each sequence of maps $\{\gamma_k^+ : \mathbb{S} \to \partial D_k^+\}$, $\{\gamma_k^- : \mathbb{S} \to \partial D_k^-\}$ converges in the spherical metric. By construction, $\gamma_{k+1}^+|_{\mathbb{S}\setminus T_k} = \gamma_k^+|_{\mathbb{S}\setminus T_k}$, and for any component W of T_k, $\gamma_{k+1}^+(W)$ and $\gamma_k^+(W)$ are contained in the same component of $\bigcap_{0\leqslant j\leqslant k} f_\lambda^{-j}(S_{s_j}^v \cup S_{-s_j}^v)$. Because the spherical metric and the hyperbolic metric are comparable in any compact subset of Υ_λ, we conclude by Lemma 2.3 that

$$\max_{t\in\mathbb{S}} \operatorname{dist}_{\overline{\mathbb{C}}}\big(\gamma_{k+1}^+(t), \gamma_k^+(t)\big) = \mathcal{O}\big(\delta^k\big),$$

where $\operatorname{dist}_{\overline{\mathbb{C}}}$ is the spherical metric and $\delta \in (0,1)$ is a constant. Thus, the sequence $\{\gamma_k^+\}$ has a limit map $\gamma_\infty^+ : \mathbb{S} \to \partial D_\infty^+$ that is continuous and surjective. Similarly, the sequence $\{\gamma_k^-\}$ also has a limit map $\gamma_\infty^- : \mathbb{S} \to \partial D_\infty^-$ that is continuous and surjective. The limit maps γ_∞^+ and γ_∞^- satisfy $\gamma_\infty^+|_{\mathbb{S}\setminus T_\infty} = \gamma_\infty^-|_{\mathbb{S}\setminus T_\infty}$. By continuity, γ_∞^+ and γ_∞^- are identical on $\mathbb{S}$. This implies that $\partial D_\infty^+ = \partial D_\infty^- = \Omega_\lambda^\theta$ and Ω_λ^θ is locally connected.

To finish, we show that Ω_λ^θ is a Jordan curve following the idea in [24]. Let $\Phi : \mathbb{D} \to D_\infty^+$ be a Riemann mapping. Because ∂D_∞^+ is locally connected, Φ has an extension from $\overline{\mathbb{D}}$ to $\overline{D_\infty^+}$. If two distinct radial segments $\Phi((0,1)e^{2\pi i\theta_1})$ and $\Phi((0,1)e^{2\pi i\theta_2})$ converge on the same point p, then the Jordan curve $\Phi((0,1)e^{2\pi i\theta_1}) \cup \Phi((0,1)e^{2\pi i\theta_2}) \cup \{\Phi(0), p\}$ separates a section of the boundary ∂D_∞^+ from D_∞^-. But this is a contradiction because D_∞^+ and D_∞^- share a common boundary. □

Proposition 3.4. *For $\lambda \in \mathcal{H}$ and $\theta \in \Theta$, all periodic points on $\Omega_\lambda^\theta \cap J(f_\lambda)$ are repulsive.*

Proof. Suppose $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots)$. Let $z \in \Omega_\lambda^\theta \cap J(f_\lambda)$ be a periodic point with period p. The itinerary of z is then of the form $(\overline{a_0, a_1, \ldots, a_{p-1}})$, where $a_j \in \{\pm s_j\}$ for $0 \leqslant j \leqslant p-1$. Let $a_k = a_{k \bmod p}$ for $k \geqslant 0$ and $S_{a_0\cdots a_s}^v = \bigcap_{0\leqslant k\leqslant s} f_\lambda^{-k}(S_{a_k}^v)$. By Lemma 2.3, the hyperbolic diameter of $S_{a_0\cdots a_s}^v$ is $\mathcal{O}(\delta^s)$ when s is large. We can therefore choose an N sufficiently large that $f_\lambda^p : \operatorname{int}(S_{a_0\cdots a_N}^v) \to \operatorname{int}(S_{a_p\cdots a_N}^v) = \operatorname{int}(S_{a_0\cdots a_{N-p}}^v)$ is a conformal map. Because $z \in \operatorname{int}(S_{a_0\cdots a_N}^v) \subset S_{a_0\cdots a_N}^v \subset \operatorname{int}(S_{a_0\cdots a_{N-p}}^v)$, we conclude $|(f_\lambda^p)'(z)| > 1$ by the Schwarz Lemma. Thus, z is a repelling periodic point. □

Proposition 3.2 tells us the combinatorial structure of the cut ray Ω_λ^θ. The following proposition shows that the iterated preimages of Ω_λ^θ have the same combinatorial structure as Ω_λ^θ provided that Ω_λ^θ does not meet the critical orbit.

Proposition 3.5. *For $\lambda \in \mathcal{H}$ and $\theta \in \Theta$, suppose the cut ray Ω_λ^θ does not meet the critical orbit. Then, for any $\alpha \in \bigcup_{k\geqslant 0}\tau^{-k}(\theta)$, there is a unique ray ω_λ^α such that:*

1. ω_λ^α *is a continuous curve connecting* 0 *with* ∞.
2. $\omega_\lambda^{\alpha+1/2} = -\omega_\lambda^\alpha$.
3. $f_\lambda(\omega_\lambda^\alpha) = \omega_\lambda^{\tau(\alpha)} \cup \omega_\lambda^{\tau(\alpha)+1/2}$.
4. $\omega_\lambda^\alpha \cap B_\lambda = R_\lambda(\alpha) \cup \{\infty\}$.

For this reason, we still call ω_λ^α a full ray of angle α and $\Omega_\lambda^\alpha = \omega_\lambda^\alpha \cup \omega_\lambda^{\alpha+1/2}$ a cut ray of angle α (or $\alpha + \frac{1}{2}$).

Proof. The proof is based on an inductive argument. Suppose $\alpha \in \bigcup_{k\geqslant 0} \tau^{-k}(\theta)$ is an angle such that the full ray ω_λ^α and the cut ray Ω_λ^α satisfy 1, 2, 3, 4. Then, for $\beta \in \tau^{-1}(\alpha)$, we define ω_λ^β by lifting Ω_λ^α in the following way:

$$f_\lambda\big(\omega_\lambda^\beta\big) = \Omega_\lambda^\alpha, \qquad \omega_\lambda^\beta \cap B_\lambda = R_\lambda(\beta) \cup \{\infty\}.$$

The ray ω_λ^β is unique because we require $\omega_\lambda^\beta \cap B_\lambda = R_\lambda(\beta) \cup \{\infty\}$. Also, by uniqueness of lifting maps, we conclude $\omega_\lambda^{\beta+\frac{1}{2}} = -\omega_\lambda^\beta$ by the fact $R_\lambda(\beta + \frac{1}{2}) = -R_\lambda(\beta)$ and $\Omega_\lambda^\alpha = -\Omega_\lambda^\alpha$.

In the following, we show that ω_λ^β connects ∞ and 0. If not, then ω_λ^β must be a curve connecting ∞ with itself, hence a Jordan curve. This implies that ω_λ^β does not meet 0. Because $\Omega_\lambda^\alpha = -\Omega_\lambda^\alpha$, all curves in the set $\mathcal{C} = \{e^{k\pi i/n}\omega_\lambda^\beta,\ H_\lambda(e^{k\pi i/n}\omega_\lambda^\beta);\ 0 \leqslant k < 2n\}$ are preimages of Ω_λ^α, where $H_\lambda(z) = \sqrt[n]{\lambda}/z$. Because Ω_λ^α does not meet the critical orbit, we conclude that for any $\gamma_1, \gamma_2 \in \mathcal{C}$ with $\gamma_1 \neq \gamma_2$, γ_1 and γ_2 are disjoint outside $\{0, \infty\}$. This means $\#\mathcal{C} = 4n$. However, this is a contradiction because the degree of f_λ is $2n$. $\square$

Recall that for any $\theta \in \Theta$ with itinerary $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots)$, the cut ray Ω_λ^θ contains at least two points, 0 and ∞, and $\Omega_\lambda^\theta \setminus \{0, \infty\}$ is contained in the interior of $S_{s_0} \cup S_{-s_0}$. Now, given two angles $\alpha, \beta \in \Theta$ with $\Omega_\lambda^\alpha \neq \Omega_\lambda^\beta$, suppose $\mathbf{s}(\alpha) = (s_0^\alpha, s_1^\alpha, s_2^\alpha, \ldots)$, $\mathbf{s}(\beta) = (s_0^\beta, s_1^\beta, s_2^\beta, \ldots)$. Let $\mathbf{J}(\alpha, \beta)$ be the first integer $k \geqslant 0$ such that $|s_k^\alpha| \neq |s_k^\beta|$. Note that the intersection $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta$ consists of at least two points 0 and ∞. Furthermore, if $\mathbf{J}(\alpha, \beta) = 0$, then $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta = \{0, \infty\}$. The following proposition tells us the number of intersection points in the general case.

Proposition 3.6. *Let* $\alpha, \beta \in \Theta$ *with* $\Omega_\lambda^\alpha \neq \Omega_\lambda^\beta$*; then, the intersection* $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta$ *consists of* $2^{\mathbf{J}(\alpha,\beta)+1}$ *points.*

Proof. We consider the orbit of $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta$ under f_λ:

$$\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta \to \Omega_\lambda^{\tau(\alpha)} \cap \Omega_\lambda^{\tau(\beta)} \to \cdots \to \Omega_\lambda^{\tau^{\mathbf{J}(\alpha,\beta)}(\alpha)} \cap \Omega_\lambda^{\tau^{\mathbf{J}(\alpha,\beta)}(\beta)}.$$

Note that for any $0 \leqslant k \leqslant \mathbf{J}(\alpha, \beta) - 1$, $f_\lambda : \Omega_\lambda^{\tau^k(\alpha)} \cap \Omega_\lambda^{\tau^k(\beta)} \to \Omega_\lambda^{\tau^{k+1}(\alpha)} \cap \Omega_\lambda^{\tau^{k+1}(\beta)}$ is a two-to-one map; thus, we have

$$\#\big(\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta\big) = 2\#\big(\Omega_\lambda^{\tau(\alpha)} \cap \Omega_\lambda^{\tau(\beta)}\big) = \cdots = 2^{\mathbf{J}(\alpha,\beta)}\#\big(\Omega_\lambda^{\tau^{\mathbf{J}(\alpha,\beta)}(\alpha)} \cap \Omega_\lambda^{\tau^{\mathbf{J}(\alpha,\beta)}(\beta)}\big) = 2^{\mathbf{J}(\alpha,\beta)+1}. \qquad \square$$

Remark 3.3. From the proof of Proposition 3.6, we know that any two distinct cut rays Ω_λ^α and Ω_λ^β intersect at the preimages of ∞. More precisely, $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta \subset \bigcup_{0\leqslant k\leqslant \mathbf{J}(\alpha,\beta)+1} f_\lambda^{-k}(\infty)$, and for $2 \leqslant k \leqslant \mathbf{J}(\alpha,\beta)+1$, the intersection $\Omega_\lambda^\alpha \cap \Omega_\lambda^\beta \cap f_\lambda^{-(k-1)}(0)$ consists of 2^{k-1} points.

4. Puzzles, graphs and tableaux

4.1. The Yoccoz puzzle

Let $X_\lambda = \bar{\mathbb{C}} \setminus \{z \in B_\lambda;\ G_\lambda(z) \geqslant 1\} = \mathbf{V}(1)$. Given N periodic angles $\theta_1, \dots, \theta_N$ that lie in different periodic cycles of Θ, let

$$g_\lambda(\theta_1, \dots, \theta_N) = \bigcup_{k\geqslant 0} \big(\Omega_\lambda^{\tau^k(\theta_1)} \cup \dots \cup \Omega_\lambda^{\tau^k(\theta_N)}\big).$$

Obviously, $g_\lambda(\theta_1, \dots, \theta_N)$ is f_λ-invariant. The graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$ generated by $\theta_1, \dots, \theta_N$ is defined as follows:

$$\mathbf{G}_\lambda(\theta_1, \dots, \theta_N) = \partial X_\lambda \cup \big(X_\lambda \cap g_\lambda(\theta_1, \dots, \theta_N)\big).$$

The Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$ is constructed in the following way. The Yoccoz puzzle of depth zero consists of all connected components of $X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$, and each component is called a puzzle piece of depth zero. The Yoccoz puzzle of greater depth can be constructed by induction as follows: if $P_d^{(1)}, \dots, P_d^{(m)}$ are the puzzle pieces of depth d, then the connected components of the set $f_\lambda^{-1}(P_d^{(i)})$ are the puzzle pieces $P_{d+1}^{(j)}$ of depth $d+1$. One can verify that the puzzle pieces of depth d consist of all connected components of $f_\lambda^{-d}(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1, \dots, \theta_N))$ and that each puzzle piece is a disk.

In applying the Yoccoz puzzle theory, we should avoid a situation in which the critical orbits touch the set $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$. If the critical orbits touch the graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$, we say the graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$ is touchable. In this case, we cannot find a sequence of shrinking puzzle pieces such that each piece contains a critical point in its interior (that is to say, we cannot find a non-degenerate critical annulus that plays a crucial role in the Yoccoz puzzle theory). For this reason, because there are infinite periodic angles in Θ, we can change the N-tuple $(\theta_1, \dots, \theta_N)$ to another N-tuple $(\theta_1', \dots, \theta_N')$ to make the graph not touchable.

Let J_0 be the set of all points on the Julia set $J(f_\lambda)$ with orbits that eventually meet the graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$. Then $J_0 = \bigcup_{k\geqslant 0} f_\lambda^{-k}(\mathbf{G}_\lambda(\theta_1, \dots, \theta_N) \cap J(f_\lambda))$. For any $z \in \bar{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, there is a unique sequence of puzzle pieces $P_0(z) \supset P_1(z) \supset P_2(z) \supset \cdots$ that contain z. By Proposition 3.4, if f_λ has a non-repelling cycle in $\mathbb{C}$, say $\mathcal{C} = \{z, f_\lambda(z), \dots, f_\lambda^p(z) = z\}$, then this cycle must avoid the graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$. This implies that $\mathcal{C} \subset \bar{\mathbb{C}} \setminus (A_\lambda \cup J_0)$. Thus, for any $d \geqslant 0$ and any $x \in \mathcal{C}$, the puzzle piece $P_d(x)$ is well defined.

Lemma 4.1. *Suppose the graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$ is not touchable, then for any $z \in \bar{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, the puzzle pieces satisfy:*

$$-P_0(z) = P_0(-z), \qquad \omega P_d(z) = P_d(\omega z), \qquad \omega^{2n} = 1, \quad d \geqslant 1.$$

Proof. By the definition of the graph $\mathbf{G}_\lambda(\theta_1,\ldots,\theta_N)$ and the symmetry of the Green function $G_\lambda : A_\lambda \to (0,+\infty]$ (see Lemma 2.1), we have $X_\lambda \setminus \mathbf{G}_\lambda(\theta_1,\ldots,\theta_N) = -X_\lambda \setminus \mathbf{G}_\lambda(\theta_1,\ldots,\theta_N)$. Thus $-P_0(z) = P_0(-z)$. Suppose that for some $d \geqslant 0$,

$$f_\lambda^{-d}\big(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1,\ldots,\theta_N)\big) = -f_\lambda^{-d}\big(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1,\ldots,\theta_N)\big).$$

Because $f_\lambda(\omega z) = \pm f_\lambda(z)$ and $G_\lambda(\omega z) = G_\lambda(z)$, we have $f_\lambda(z) \in f_\lambda^{-d}(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1,\ldots,\theta_N))$ if and only if $f_\lambda(\omega z) \in f_\lambda^{-d}(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1,\ldots,\theta_N))$. Thus

$$f_\lambda^{-(d+1)}\big(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1,\ldots,\theta_N)\big) = \omega f_\lambda^{-(d+1)}\big(X_\lambda \setminus \mathbf{G}_\lambda(\theta_1,\ldots,\theta_N)\big).$$

The conclusion follows by induction. □

Lemma 4.2. *Suppose the graph $\mathbf{G}_\lambda(\theta_1,\ldots,\theta_N)$ is not touchable, then for any $d \geqslant 0$ and any puzzle piece P_d of depth d, the intersection $\bar{P}_d \cap J(f_\lambda)$ is connected.*

Proof. It is equivalent to prove that every connected component of $\bar{\mathbb{C}} \setminus (\bar{P}_d \cap J(f_\lambda))$ is simply connected. Because the Julia set $J(f_\lambda)$ is connected, every component of $\bar{\mathbb{C}} \setminus (\bar{P}_d \cap J(f_\lambda))$ that lies inside P_d is simply connected. Therefore, we only need to consider the components of $\bar{\mathbb{C}} \setminus (\bar{P}_d \cap J(f_\lambda))$ that intersect with ∂P_d. Note that the puzzle piece P_d is bounded by finitely many cut rays, say $\Omega_\lambda^{\beta_1},\ldots,\Omega_\lambda^{\beta_s}$, together with finitely many equipotential curves $\mathbf{e}(U_1,v),\ldots,\mathbf{e}(U_t,v)$. By the structure of cut rays (Proposition 3.2), there is exactly one component of $\bar{\mathbb{C}} \setminus (\bar{P}_d \cap J(f_\lambda))$ that intersects with the boundary ∂P_d. This component is the union of $\bar{\mathbb{C}} \setminus \bar{P}_d$ and countably many Fatou components that intersect with the cut rays $\Omega_\lambda^{\beta_1},\ldots,\Omega_\lambda^{\beta_s}$. Thus, it is also simply connected. □

4.2. Admissible graphs

Given the point $z \in \bar{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, the difference set $A_d(z) = P_d(z) \setminus \overline{P_{d+1}(z)}$ is an annulus, either degenerate or of positive modulus. Here, d is called the depth of $A_d(z)$. For $d \geqslant 1$ and $c \in C_\lambda$, the annulus $A_d(z)$ is called off-critical, c-critical or c-semi-critical if $P_d(z)$ contains no critical points, $P_{d+1}(z)$ contains the critical point c or $A_d(z)$ contains the critical point c, respectively.

Because the critical annuli play a crucial role in our discussion, we will devote ourselves to finding a graph such that with respect to the Yoccoz puzzle induced by such a graph, the critical annulus $A_d(c)$ is non-degenerate for some $d \geqslant 1$. By Lemma 4.1, if some critical annulus $A_d(c)$ of depth $d \geqslant 1$ is non-degenerate, then all critical annuli of the same depth are non-degenerate. The graph that satisfies this property is of special interest.

Definition 4.1. We say the graph $\mathbf{G}_\lambda(\theta_1,\ldots,\theta_N)$ is admissible if it is not touchable and if with respect to the Yoccoz puzzle induced by $\mathbf{G}_\lambda(\theta_1,\ldots,\theta_N)$ there exists a non-degenerate critical annulus $A_d(c)$ for some critical point $c \in C_\lambda$ and some depth $d \geqslant 1$. Otherwise, we say the graph $\mathbf{G}_\lambda(\theta_1,\ldots,\theta_N)$ is non-admissible.

By definition, a non-admissible graph either is touchable or contains no non-degenerate critical annulus of depth greater than one with respect to its induced Yoccoz puzzle. In the definition

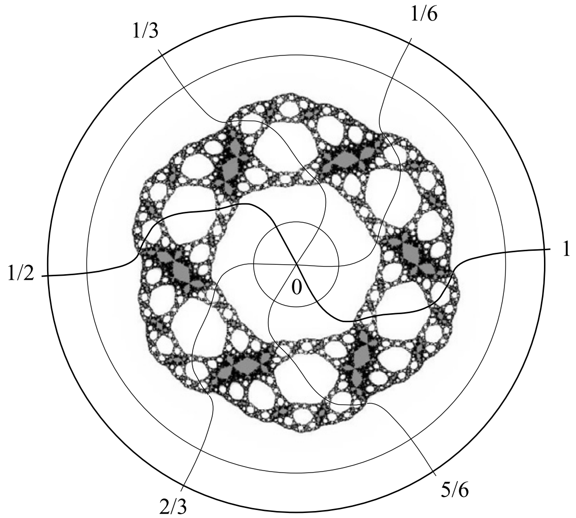

Fig. 6. A graph with Yoccoz puzzle to depth one ($n = 3$ and $\mathbf{G}_\lambda = \mathbf{G}_\lambda(1/2)$).

of an admissible graph, we require that the critical annulus $A_d(c)$ is non-degenerate for some depth $d \geqslant 1$ rather than $d = 0$ because the puzzle pieces of depth zero have only two-fold symmetry and the puzzle pieces of depth greater than zero have $2n$-fold symmetry (see Lemma 4.1).

The following remark tells us that a graph may be non-admissible in some cases.

Remark 4.1. There exist non-admissible graphs. For example, for any $n \geqslant 3$, suppose f_λ is 1-renormalizable at c_0 (see Section 5 for definition). Then, the graph $\mathbf{G}_\lambda(1)$ is non-admissible because $A_d(c_0)$ is degenerate for all depths $d \geqslant 1$ (see Fig. 6). One should note that $A_0(c_1)$ is non-degenerate and $A_d(c_1) = e^{\pi i/3} A_d(c_0)$ is degenerate for all $d \geqslant 1$.

However, even if non-admissible graphs exist, we can always find an admissible graph based on an elaborate choice. The aim of this section is to prove the existence of admissible graphs for $n \geqslant 3$.

Proposition 4.1. *For any $n \geqslant 3$ and any $\lambda \in \mathcal{H}$, if f_λ is not critically finite, then there always exists an admissible graph.*

The proof is divided into three lemmas: Lemma 4.3, Lemma 4.4 and Lemma 4.5. In fact, these lemmas enable us to prove much more: when $n \geqslant 5$, there always exist infinitely many admissible graphs $f_\lambda^{k+1}(\sqrt[2n]{\lambda}) = f_\lambda^k(\sqrt[2n]{\lambda})$ or $f_\lambda^{k+2}(\sqrt[2n]{\lambda}) = f_\lambda^k(\sqrt[2n]{\lambda})$ for some $k \geqslant 1$.

Lemma 4.3. *When $n = 3$, there exists an admissible graph except when the critical orbit of f_λ eventually lands at a repelling cycle of period one or two. More precisely,*

1. *If neither $\mathbf{G}_\lambda(1/4)$ nor $\mathbf{G}_\lambda(1/2)$ is touchable, then at least one of the graphs $\mathbf{G}_\lambda(1/4)$, $\mathbf{G}_\lambda(1/2)$, $\mathbf{G}_\lambda(1/4, 1/2)$ is admissible.*
2. *If $\mathbf{G}_\lambda(1/2)$ is touchable, then either $\mathbf{G}_\lambda(1/4)$ is admissible or the critical orbit of f_λ eventually lands at a repelling cycle of period two.*

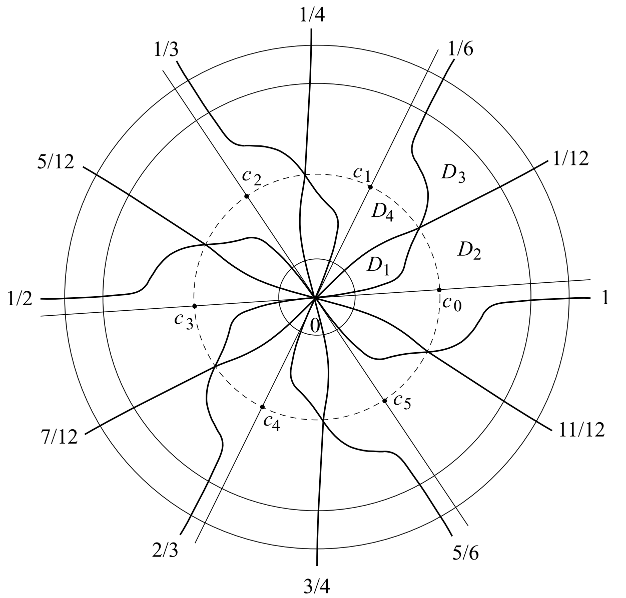

Fig. 7. Candidates for admissible graph when $n = 3$.

3. *If* $\mathbf{G}_\lambda(1/4)$ *is touchable, then either* $\mathbf{G}_\lambda(1/2)$ *is admissible or the critical orbit of* f_λ *eventually lands at a repelling fixed point.*

Proof. First, note that

$$f_\lambda^{-1}\big(\Omega_\lambda^{1/4}\big) = \Omega_\lambda^{1/12} \cup \Omega_\lambda^{1/4} \cup \Omega_\lambda^{5/12}, \qquad f_\lambda^{-1}\big(\Omega_\lambda^{1/2}\big) = \Omega_\lambda^{1/6} \cup \Omega_\lambda^{1/3} \cup \Omega_\lambda^{1/2}.$$

1. In this case, the full rays $\omega_\lambda^{1/12}$ and $\omega_\lambda^{1/6}$ decompose S_0 into four domains: D_1, D_2, D_3 and D_4 (see Fig. 7). If neither $\mathbf{G}_\lambda(1/4)$ nor $\mathbf{G}_\lambda(1/2)$ is touchable, then the critical orbit has no intersection with $\Omega_\lambda^{1/4} \cup \Omega_\lambda^{1/2}$.

We consider the location of the critical value v_λ^+; there are four possibilities:

Case 1. $v_\lambda^+ \in D_1$. In this case, the annulus $A_0(v_\lambda^+) = P_0(v_\lambda^+) \setminus \overline{P_1(v_\lambda^+)}$ is non-degenerate with respect to the Yoccoz puzzle as induced by either of the graphs $\mathbf{G}_\lambda(1/4)$, $\mathbf{G}_\lambda(1/2)$ and $\mathbf{G}_\lambda(1/4, 1/2)$. It turns out that the critical annulus $A_1(c)$ is non-degenerate for all $c \in C_\lambda$. Thus, in this case, all the graphs $\mathbf{G}_\lambda(1/4)$, $\mathbf{G}_\lambda(1/2)$, $\mathbf{G}_\lambda(1/4, 1/2)$ are admissible.

Case 2. $v_\lambda^+ \in D_2$. The annulus $A_0(v_\lambda^+) = P_0(v_\lambda^+) \setminus \overline{P_1(v_\lambda^+)}$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(1/4)$. Therefore, all critical annuli $A_1(c)$ are non-degenerate. Thus, the graph $\mathbf{G}_\lambda(1/4)$ is admissible.

Case 3. $v_\lambda^+ \in D_3$. The annulus $A_0(v_\lambda^+)$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(1/4, 1/2)$. Therefore, all critical annuli $A_1(c)$ are non-degenerate, and the graph $\mathbf{G}_\lambda(1/4, 1/2)$ is admissible.

Case 4. $v_\lambda^+ \in D_4$. Based on an argument similar to that used above, we conclude that the graph $\mathbf{G}_\lambda(1/2)$ is admissible.

2. In this case, the graph $\mathbf{G}_\lambda(1/4)$ is necessarily untouchable. First, note that the cut ray $\Omega_\lambda^{5/12}$ decomposes $\Omega_\lambda^{1/2}$ into four parts: $\Omega_\lambda^{1/2}(2,2)$, $\Omega_\lambda^{1/2}(2,-2)$, $\Omega_\lambda^{1/2}(-2,2)$ and $\Omega_\lambda^{1/2}(-2,-2)$, where

$$\Omega_\lambda^{1/2}(\epsilon_0,\epsilon_1) = \overline{\{z \in \Omega_\lambda^{1/2} \setminus O_\lambda;\ \mathbf{s}_\lambda(z) = (\epsilon_0, \epsilon_1, \pm 2, \pm 2, \ldots)\}}, \quad \epsilon_0, \epsilon_1 = \pm 2.$$

Moreover, for any $z \in (\Omega_\lambda^{1/2}(2,2) \cup \Omega_\lambda^{1/2}(-2,-2)) \cap J(f_\lambda)$, the annulus $A_0(z)$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(1/4)$.

Because $\mathbf{G}_\lambda(1/2)$ is touchable, there exist an integer $p \geqslant 1$ and a critical point $c \in C_\lambda$ such that $f_\lambda^p(c) \in \Omega_\lambda^{1/2}$. Consider the itinerary of $f_\lambda^p(c)$, say $\mathbf{s}_\lambda(f_\lambda^p(c)) = (s_0, s_1, s_2, \ldots)$. There are two possibilities:

Case 1. There is an integer $n \geqslant 0$ such that $(s_n, s_{n+1}) = (2,2)$ or $(-2,-2)$. In this case, $f_\lambda^{n+p}(c) \in (\Omega_\lambda^{1/2}(2,2) \cup \Omega_\lambda^{1/2}(-2,-2)) \cap J(f_\lambda)$; thus, the annulus $A_0(f_\lambda^{n+p}(c))$ is non-degenerate. It turns out that the critical annulus $A_{n+p}(c)$ is non-degenerate. Therefore, the graph $\mathbf{G}_\lambda(1/4)$ is admissible.

Case 2. For any integer $n \geqslant 0$, $(s_n, s_{n+1}) = (2,-2)$ or $(-2,2)$. In this case, either $\mathbf{s}_\lambda(f_\lambda^p(c)) = (2,-2,2,-2,\ldots) = (\overline{2,-2})$ or $\mathbf{s}_\lambda(f_\lambda^p(c)) = (-2,2,-2,2,\ldots) = (\overline{-2,2})$. By Proposition 3.4, $f_\lambda^p(c)$ lies in a repelling cycle of period two.

3. The proof is similar to the proof of 2. In this case, the graph $\mathbf{G}_\lambda(1/2)$ is necessarily untouchable. First, note that the cut ray $\Omega_\lambda^{1/3}$ decomposes $\Omega_\lambda^{1/4}$ into four parts: $\Omega_\lambda^{1/4}(1,-1)$, $\Omega_\lambda^{1/4}(1,1)$, $\Omega_\lambda^{1/4}(-1,-1)$ and $\Omega_\lambda^{1/4}(-1,1)$, where

$$\Omega_\lambda^{1/4}(\epsilon_0,\epsilon_1) = \overline{\{z \in \Omega_\lambda^{1/4} \setminus O_\lambda;\ \mathbf{s}_\lambda(z) = (\epsilon_0, \epsilon_1, \pm 1, \pm 1, \ldots)\}}, \quad \epsilon_0, \epsilon_1 = \pm 1.$$

Moreover, for any $z \in (\Omega_\lambda^{1/4}(1,-1) \cup \Omega_\lambda^{1/4}(-1,1)) \cap J(f_\lambda)$, the annulus $A_0(z)$ is non-degenerate with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(1/2)$.

Because $\mathbf{G}_\lambda(1/4)$ is touchable, there are an integer $p \geqslant 1$ and a critical point $c \in C_\lambda$ such that $f_\lambda^p(c) \in \Omega_\lambda^{1/4}$. Consider the itinerary of $f_\lambda^p(c)$, say $\mathbf{s}_\lambda(f_\lambda^p(c)) = (s_0, s_1, s_2, \ldots)$. There are two possibilities:

Case 1. There is an integer $n \geqslant 0$ such that $(s_n, s_{n+1}) = (-1,1)$ or $(1,-1)$. In this case, $f_\lambda^{n+p}(c) \in (\Omega_\lambda^{1/4}(1,-1) \cup \Omega_\lambda^{1/4}(-1,1)) \cap J(f_\lambda)$; thus, the annulus $A_0(f_\lambda^{n+p}(c))$ is non-degenerate. It turns out that the critical annulus $A_{n+p}(c)$ is non-degenerate. Therefore, the graph $\mathbf{G}_\lambda(1/2)$ is admissible.

Case 2. For any integer $n \geqslant 0$, $(s_n, s_{n+1}) = (1,1)$ or $(-1,-1)$. In this case, either $\mathbf{s}_\lambda(f_\lambda^p(c)) = (1,1,\ldots) = (\overline{1})$ or $\mathbf{s}_\lambda(f_\lambda^p(c)) = (-1,-1,\ldots) = (\overline{-1})$. By Proposition 3.4, $f_\lambda^p(c)$ is a repelling fixed point. $\square$

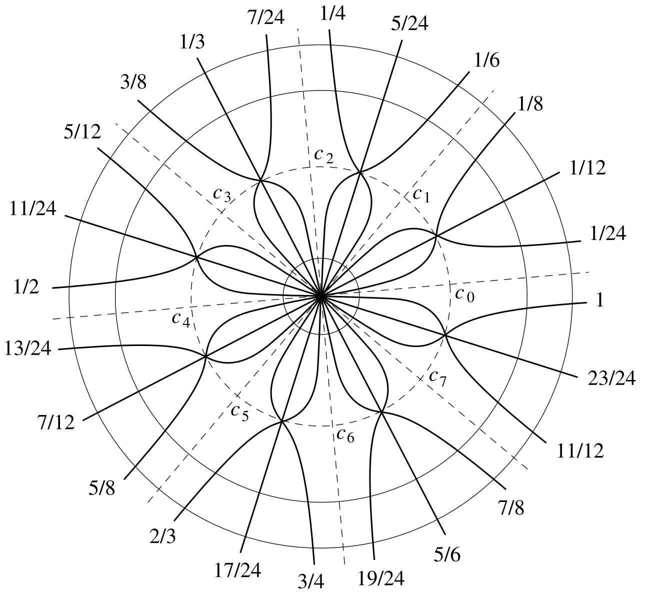

Fig. 8. Candidates for admissible graph when $n = 4$.

Lemma 4.4. *When $n = 4$, if $\mathbf{G}_\lambda(1/3)$ is not touchable, then $\mathbf{G}_\lambda(1/3)$ is admissible; if $\mathbf{G}_\lambda(1/3)$ is touchable, then $\mathbf{G}_\lambda(2/3, 1)$ is admissible.*

Proof. First, note that $\mathbf{s}(1/3) = (2, 2, \ldots) = (\bar{2})$, $\mathbf{s}(2/3) = (-1, -1, \ldots) = (\overline{-1})$ and $\mathbf{s}(1) = (-3, -3, \ldots) = (\overline{-3})$. Thus, $\Omega_\lambda^{1/3} \subset S_2 \cup S_{-2}$, $\Omega_\lambda^{2/3} \subset S_1 \cup S_{-1}$ and $\Omega_\lambda^{1} \subset S_3 \cup S_{-3}$ (see Fig. 8). It is easy to verify

$$f_\lambda^{-1}\big(\Omega_\lambda^{1/3}\big) = \Omega_\lambda^{1/12} \cup \Omega_\lambda^{5/24} \cup \Omega_\lambda^{1/3} \cup \Omega_\lambda^{11/24}.$$

If the graph $\mathbf{G}_\lambda(1/3)$ is not touchable, then the critical orbit has no intersection with $\Omega_\lambda^{1/3}$. With respect to the Yoccoz puzzle induced by $\mathbf{G}_\lambda(1/3)$, the puzzle piece $P_1(v_\lambda^+)$ is a subset of the domain bounded by $\omega_\lambda^{5/24}$ and $\omega_\lambda^{23/24}$ together with the equipotential curves $\mathbf{e}(B_\lambda, 1/n)$ and $\mathbf{e}(T_\lambda, 1/n)$. Thus, the annulus $A_0(v_\lambda^+)$ is non-degenerate. It turns out that all critical annuli $A_1(c)$ are non-degenerate. Therefore, the graph $\mathbf{G}_\lambda(1/3)$ is admissible.

If the graph $\mathbf{G}_\lambda(1/3)$ is touchable, then there exist an integer $p \geqslant 1$ and a critical point $c \in C_\lambda$ such that $f_\lambda^p(c) \in \Omega_\lambda^{1/3}$. Note that the preimage of $\Omega_\lambda^{2/3}$ that lies in $S_2 \cup S_{-2}$ is $\Omega_\lambda^{7/24}$ and the preimage of Ω_λ^{1} that lies in $S_2 \cup S_{-2}$ is $\Omega_\lambda^{3/8}$. In this case, with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(2/3, 1)$, the puzzle piece $P_1(f_\lambda^p(c))$ is bounded by $\Omega_\lambda^{7/24}$ and $\Omega_\lambda^{3/8}$; thus, the annulus $A_0(f_\lambda^p(c))$ is non-degenerate. It follows that all critical annuli $A_p(c)$ are non-degenerate, and the graph $\mathbf{G}_\lambda(2/3, 1)$ is admissible. $\square$

In the following, we will consider the case when $n \geqslant 5$. Let

$$\hat{\Theta} = \bigcap_{j \geqslant 0} \tau^{-j}\Bigg(\bigcup_{2 \leqslant k \leqslant n-2} (\Theta_k \cup \Theta_{-k})\Bigg)$$

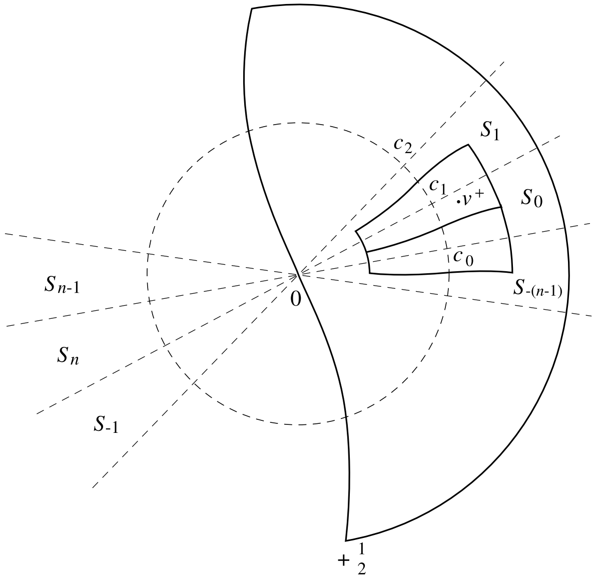

Fig. 9. Candidates for admissible graph when $n \geqslant 5$.

be the set of all angles in Θ whose orbits remain in $\bigcup_{2\leqslant k\leqslant n-2}(\Theta_k \cup \Theta_{-k})$ under all iterations of τ, and let $\hat{\Theta}_{per}$ be the set of all periodic angles in $\hat{\Theta}$. Based on a similar argument as for Lemma 3.2, we can show that $\hat{\Theta}_{per}$ is a dense subset of $\hat{\Theta}$. By Lemma 3.1, one can check that the set $\hat{\Theta}_{per}$ can be written as

$$\hat{\Theta}_{per} = \bigcup_{p\geqslant 1} \bigl\{\kappa(\mathbf{s});\ \mathbf{s} = (\overline{s_0,\dots,s_{p-1}}) \in \Sigma_0 \text{ and } s_0,\dots,s_{p-1} \in \bigl\{\pm 2,\dots,\pm(n-2)\bigr\}\bigr\}$$

and that any angle $\theta \in \hat{\Theta}_{per}$ is of the form

$$\theta = \frac{1}{2}\left(\frac{\chi(s_0)}{n} + \frac{|s_0|}{n(n^p-1)} + \frac{n^p}{n^p-1}\sum_{1\leqslant k<p}\frac{|s_k|}{n^{k+1}}\right).$$

Lemma 4.5. *When $n \geqslant 5$, there are infinitely many periodic angles $\theta \in \Theta$ such that the graph $\mathbf{G}_\lambda(\theta)$ is admissible.*

Proof. We can choose an angle $\theta \in \hat{\Theta}_{per}$ such that the critical orbit avoids the graph $\mathbf{G}_\lambda(\theta)$. (Note that there are infinitely many such choices of angle θ.) When $n \geqslant 5$, the set $\bigcup_{j\geqslant 0} \Omega_\lambda^{\tau^j(\theta)} - \{0,\infty\}$ lies outside $S_1 \cup S_0 \cup S_{-(n-1)}$ (see Fig. 9). Then, with respect to the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(\theta)$, $\overline{P_1(v_\lambda^+)}$ is contained in the interior of $S_1 \cup S_0 \cup S_{-(n-1)}$ and is a proper subset of $P_0(v_\lambda^+)$. Because $f_\lambda(P_2(c_0)) = P_1(v_\lambda^+)$ and $f_\lambda(P_1(c_0)) = P_0(v_\lambda^+)$, we know that $A_1(c_0) = P_1(c_0) \setminus \overline{P_2(c_0)}$ is non-degenerate. Thus, the graph $\mathbf{G}_\lambda(\theta)$ is admissible. $\square$

In the remainder of this section, we prove an important property of the cut rays that are used to generate admissible graphs. Let

$$\Theta_{ad} = \begin{cases} \{\frac{1}{4}, \frac{1}{2}\}, & n=3, \\ \{\frac{1}{3}, \frac{2}{3}, 1\}, & n=4, \\ \hat{\Theta}_{per}, & n \geqslant 5. \end{cases}$$

Note that for any admissible graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$ constructed by Lemma 4.3, Lemma 4.4 and Lemma 4.5, $\{\theta_1, \dots, \theta_N\} \subset \Theta_{ad}$. In the following, we will prove

Proposition 4.2. *For any $\theta \in \Theta_{ad}$, the intersection $\Omega_\lambda^\theta \cap \partial B_\lambda$ consists of two points.*

The proof is based on the following:

Lemma 4.6. *Suppose $\theta \in \Theta$, and θ satisfies one of the following conditions:*

C1. *There are two sequences, $\{\theta_k^+\}_{k\geqslant 1}, \{\theta_k^-\}_{k\geqslant 1} \subset \Theta$ such that for all $k \geqslant 1$, $\theta_k^- < \theta < \theta_k^+$ and $\mathbf{J}(\theta_k^+, \theta) = \mathbf{J}(\theta_k^-, \theta) \to \infty$ as $k \to \infty$.*
C2. *There is a sequence $\{\theta_k\}_{k\geqslant 1} \subset \Theta$ such that $\theta_1 < \theta_2 < \theta_3 < \cdots$ (or $\theta_1 > \theta_2 > \theta_3 > \cdots$) and $\mathbf{J}(\theta_k, \theta) = k$ for any $k \geqslant 1$.*

Then the intersection $\Omega_\lambda^\theta \cap \partial B_\lambda$ consists of two points.

Proof. 1. Suppose θ satisfies C1 and $\mathbf{s}(\theta) = (s_0, s_1, s_2, \dots)$. By Proposition 3.6, the cut rays $\Omega_\lambda^{\theta_k^+}$ and $\Omega_\lambda^{\theta_k^-}$ both intersect with Ω_λ^θ at $2^{\mathbf{J}(\theta_k^+,\theta)+1}$ points; they hence decompose Ω_λ^θ into $2^{\mathbf{J}(\theta_k^+,\theta)+1}$ parts:

$$\Omega_\lambda^\theta(\epsilon_0, \epsilon_1, \dots, \epsilon_{\mathbf{J}(\theta_k^+,\theta)}), \quad \epsilon_j = \pm s_j,\ 0 \leqslant j \leqslant \mathbf{J}\big(\theta_k^+, \theta\big).$$

Here $\Omega_\lambda^\theta(\epsilon_0, \epsilon_1, \dots, \epsilon_p) := \overline{\{z \in \Omega_\lambda^\theta \setminus O_\lambda;\ \mathbf{s}_\lambda(z) = (\epsilon_0, \epsilon_1, \dots, \epsilon_p, \pm s_{p+1}, \pm s_{p+2}, \dots)\}}$.

Based on the structure of the cut rays (Proposition 3.2) and because the angle θ satisfies condition C1, we conclude that of these $2^{\mathbf{J}(\theta_k^+,\theta)+1}$ parts, only two intersect with $\overline{B_\lambda}$: $\Omega_\lambda^\theta(s_0, s_1, \dots, s_{\mathbf{J}(\theta_k^+,\theta)})$ and $\Omega_\lambda^\theta(-s_0, (-1)^n s_1, \dots, (-1)^n s_{\mathbf{J}(\theta_k^+,\theta)})$. We should remark that here we use two cut rays $\Omega_\lambda^{\theta_k^+}, \Omega_\lambda^{\theta_k^-}$ with $\mathbf{J}(\theta_k^+, \theta) = \mathbf{J}(\theta_k^-, \theta)$ to separate the other segments of Ω_λ^θ from $\overline{B_\lambda}$ (see Fig. 10). Moreover, $\Omega_\lambda^\theta \cap \overline{B_\lambda} \subset \Omega_\lambda^\theta(s_0, s_1, \dots, s_{\mathbf{J}(\theta_k^+,\theta)}) \cup \Omega_\lambda^\theta(-s_0, (-1)^n s_1, \dots, (-1)^n s_{\mathbf{J}(\theta_k^+,\theta)})$ for any $k \geqslant 1$. It turns out that

$$\begin{aligned} \Omega_\lambda^\theta \cap \overline{B_\lambda} &\subset \bigcap_{k\geqslant 1} \big(\Omega_\lambda^\theta(s_0, s_1, \dots, s_{\mathbf{J}(\theta_k^+,\theta)}) \cup \Omega_\lambda^\theta\big(-s_0, (-1)^n s_1, \dots, (-1)^n s_{\mathbf{J}(\theta_k^+,\theta)}\big)\big) \\ &= \big\{z \in \Omega_\lambda^\theta;\ \mathbf{s}_\lambda(z) = (s_0, s_1, s_2, \dots) \text{ or } \big(-s_0, (-1)^n s_1, (-1)^n s_2, \dots\big)\big\} \\ &= \overline{R_\lambda(\theta)} \cup \overline{R_\lambda(\theta + 1/2)}. \end{aligned}$$

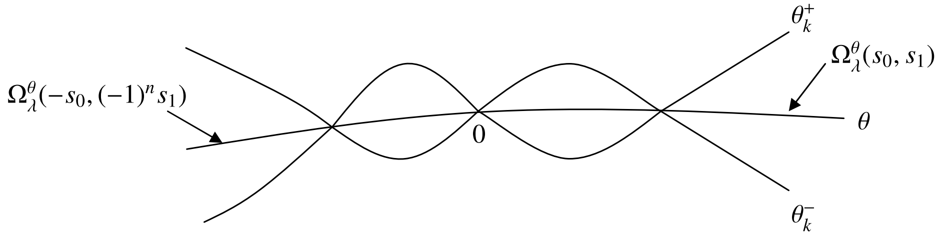

Fig. 10. Three cuts rays with angles $\theta_k^+ > \theta > \theta_k^-$. In this figure, $\mathbf{J}(\theta_k^+,\theta) = \mathbf{J}(\theta_k^-,\theta) = 1$. Exactly two segments of Ω_λ^θ intersect with $\overline{B_\lambda}$: $\Omega_\lambda^\theta(s_0,s_1)$ and $\Omega_\lambda^\theta(-s_0,(-1)^n s_1)$.

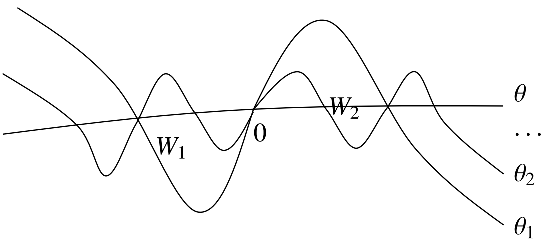

Fig. 11. Cut rays with angles $\theta_1 < \theta_2 < \cdots < \theta$, $\mathbf{J}(\theta,\theta_1) = 1, \mathbf{J}(\theta,\theta_2) = 2, \ldots$. Moreover, $\overline{B_\lambda}$ has no intersection with the bounded components W_1 and W_2 of $\bar{\mathbb{C}} \setminus (\Omega_\lambda^{\theta_1} \cup \Omega_\lambda^{\theta_2})$.

By Proposition 3.2, the intersection $\Omega_\lambda^\theta \cap \partial B_\lambda$ consists of two points. These two points are the landing points of the external rays $R_\lambda(\theta)$ and $R_\lambda(\theta+1/2)$.

2. Now we suppose that θ satisfies C2 and $\mathbf{s}(\theta) = (s_0,s_1,s_2,\ldots)$. We only prove the case when n is odd. The argument applies equally well to the case when n is even. Let $\{\theta_k\}_{k\geqslant 1} \subset \Theta$ be a sequence such that $\theta_1 < \theta_2 < \theta_3 < \cdots$ and $\mathbf{J}(\theta_k,\theta) = k$ for any $k \geqslant 1$. The following facts are straightforward:

Fact 1. Let $z \in \Omega_\lambda^\theta$. If the itinerary $\mathbf{s}_\lambda(z)$ is of the form $(\epsilon_0,\ldots,\epsilon_k,s_{k+1},s_{k+2},\ldots)$ or $(\epsilon_0,\ldots,\epsilon_k,-s_{k+1},-s_{k+2},\ldots)$ for some $k \geqslant 0$, then $\mathbf{s}_\lambda(f_\lambda^{k+1}(z)) = \pm(s_{k+1},s_{k+2},\ldots) = \mathbf{s}(\tau^{k+1}(\theta))$ or $\mathbf{s}(\tau^{k+1}(\theta)+\frac{1}{2})$. By Proposition 3.1, $f_\lambda^{k+1}(z) \in \overline{R_\lambda(\tau^{k+1}(\theta))} \cup \overline{R_\lambda(\tau^{k+1}(\theta)+\frac{1}{2})}$. Thus, z lies in the closure of some external ray or radial ray $R_U(\theta_U)$ for $U \in \mathcal{P}$.

Fact 2. For any $k > 1$, $\overline{B_\lambda}$ has no intersection with any bounded component of $\bar{\mathbb{C}} \setminus \bigcup_{1\leqslant j\leqslant k} \Omega_\lambda^{\theta_j}$; see Fig. 11. (The proof is almost immediate from Proposition 3.1.)

Fact 3. The sections of Ω_λ^θ that intersect with the unbounded component of $\bar{\mathbb{C}} \setminus \bigcup_{1\leqslant j\leqslant k} \Omega_\lambda^{\theta_j}$ are as follows:

$$\Omega_\lambda^\theta(s_0,\ldots,s_k), \qquad \Omega_\lambda^\theta(-s_0,\ldots,-s_k),$$
$$\Omega_\lambda^\theta(s_0,\ldots,s_j,-s_{j+1},\ldots,-s_k), \qquad \Omega_\lambda^\theta(-s_0,\ldots,-s_j,s_{j+1},\ldots,s_k), \quad 0 \leqslant j < k.$$

Let $\mathcal{E}_k$ be the collection of these sections.

Based on Facts 2 and 3, we have $\overline{B_\lambda} \cap \Omega_\lambda^\theta \subset \bigcup_{E\in\mathcal{E}_k} E$ for any $k > 1$. It follows that $\overline{B_\lambda} \cap \Omega_\lambda^\theta \subset \bigcap_{k>1} \bigcup_{E\in\mathcal{E}_k} E = \{z \in \Omega_\lambda^\theta, \mathbf{s}_\lambda(z)$ is of the form $\pm\mathbf{s}(\theta)$ or $\pm(s_0, s_1, \ldots, s_k, -s_{k+1}, -s_{k+2}, \ldots)$ for some $k \geqslant 0\}$.

By Fact 1, for any $z \in \overline{B_\lambda} \cap \Omega_\lambda^\theta$, either $z \in \overline{R_\lambda(\theta)} \cup \overline{R_\lambda(\theta + 1/2)}$ or there exist $U \in \mathcal{P} \setminus \{B_\lambda\}$ and an angle θ_U such that $z \in \overline{R_U(\theta_U)}$. In the following, we show that the latter is impossible. In fact, if $z \in \overline{B_\lambda} \cap \Omega_\lambda^\theta \cap \overline{R_U(\theta_U)}$, then $z \in \partial B_\lambda \cap \partial U$. Let $p \geqslant 0$ be the first integer such that $f_\lambda^p(U) = T_\lambda$.

After p iterations, we see that $f_\lambda^p(z) \in \partial B_\lambda \cap \partial T_\lambda$ and $f_\lambda^p(z)$ is the landing point of the radial ray $R_{T_\lambda}(\theta_{T_\lambda}) = f_\lambda^p(R_U(\theta_U))$. On the other hand, $f_\lambda^{p+1}(z)$ is the landing point of the external ray $R_\lambda(\theta_\lambda) = f_\lambda^{p+1}(R_U(\theta_U))$. Therefore, $f_\lambda^p(z)$ is also a landing point of some external ray $R_\lambda(\beta)$, $\beta \in \tau^{-1}(\theta_\lambda)$. Because both $R_{T_\lambda}(\theta_{T_\lambda})$ and $R_\lambda(\beta)$ land at $f_\lambda^p(z)$, and $f_\lambda(R_{T_\lambda}(\theta_{T_\lambda})) = f_\lambda(R_\lambda(\beta)) = R_\lambda(\theta_\lambda)$, $f_\lambda^p(z)$ is necessarily a critical point in C_λ.

However, the result that $f_\lambda^p(z) \in f_\lambda^p(\Omega_\lambda^\theta) \cap C_\lambda$ leads to a contradiction because for any $\alpha \in \Theta$, the cut ray Ω_λ^α avoids the critical set C_λ.

Now, we are in the situation $\overline{B_\lambda} \cap \Omega_\lambda^\theta \subset \overline{R_\lambda(\theta)} \cup \overline{R_\lambda(\theta + 1/2)}$, and the conclusion follows. $\square$

Proof of Proposition 4.2. It suffices to verify that for any $\theta \in \Theta_{ad}$, θ satisfies either C1 or C2 by Lemma 4.6.

When $n = 3$, $\mathbf{s}(1/4) = (\overline{1, -1})$, $\mathbf{s}(1/2) = (\bar{2})$. Define two sequences of angles $\{\alpha_k\}_{k\geqslant 1}$, $\{\beta_k\}_{k\geqslant 1} \subset \Theta$ such that

$$\begin{aligned}
&\mathbf{s}(\alpha_1) = (1, -2, -1, 1, -1, 1, \ldots), &&\mathbf{s}(\beta_1) = (2, 1, -1, 2, 2, 2, \ldots),\\
&\mathbf{s}(\alpha_2) = (1, -1, 2, 1, -1, 1, \ldots), &&\mathbf{s}(\beta_2) = (2, 2, 1, -1, 2, 2, \ldots),\\
&\mathbf{s}(\alpha_3) = (1, -1, 1, -2, -1, 1, \ldots), &&\mathbf{s}(\beta_3) = (2, 2, 2, 1, -1, 2, \ldots),\\
&\ldots
\end{aligned}$$

Then, $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$ and $\mathbf{J}(\alpha_k, 1/4) = k$ for any $k \geqslant 1$; $\beta_1 < \beta_2 < \beta_3 < \cdots$ and $\mathbf{J}(\beta_k, 1/2) = k$. Thus, both 1/4 and 1/2 satisfy condition C2.

When $n = 4$, $\mathbf{s}(1/3) = (\bar{2})$, $\mathbf{s}(2/3) = (\overline{-1})$, $\mathbf{s}(1) = (\overline{-3})$. Define three sequences of angles $\{\alpha_k\}_{k\geqslant 1}, \{\beta_k\}_{k\geqslant 1}, \{\gamma_k\}_{k\geqslant 1} \subset \Theta$ such that

$$\begin{aligned}
&\mathbf{s}(\alpha_1) = (2, 1, -2, 2, 2, \ldots), &&\mathbf{s}(\beta_1) = (-1, -3, -1, -1, \ldots),\\
&\mathbf{s}(\gamma_1) = (-3, -1, -3, -3, \ldots), &&\mathbf{s}(\alpha_2) = (2, 2, 1, -2, 2, \ldots),\\
&\mathbf{s}(\beta_2) = (-1, -1, -3, -1, \ldots), &&\mathbf{s}(\gamma_2) = (-3, -3, -1, -3, \ldots),\\
&\mathbf{s}(\alpha_3) = (2, 2, 2, 1, -2, \ldots), &&\mathbf{s}(\beta_3) = (-1, -1, -1, -3, \ldots),\\
&\mathbf{s}(\gamma_3) = (-3, -3, -3, -1, \ldots), &&\cdots
\end{aligned}$$

Then $\alpha_1 < \alpha_2 < \alpha_3 < \cdots$ and $\mathbf{J}(\alpha_k, 1/3) = k$; $\beta_1 > \beta_2 > \beta_3 > \cdots$ and $\mathbf{J}(\beta_k, 2/3) = k$; $\gamma_1 < \gamma_2 < \gamma_3 < \cdots$ and $\mathbf{J}(\gamma_k, 1) = k$. Thus, 1/3, 2/3, 1 all satisfy condition C2.

When $n \geqslant 5$, we can prove that for any $\theta \in \hat{\Theta}_{per}$, θ satisfies condition C1. (In fact, this is true for all $\theta \in \hat{\Theta}$.) The proof is as follows. Suppose $\mathbf{s}(\theta) = (s_0, s_1, s_2, \ldots)$. For any $k \geqslant 1$, we choose $s_k^-, s_k^+ \in \{\pm 1, \pm(n-1)\}$ and $s_{k+1}^-, s_{k+1}^+ \in \mathbb{I} \setminus \{0, n\}$ such that

(1) $|s_k^-| < |s_k| < |s_k^+|$,
(2) $(s_0, \dots, s_{k-1}, s_k^-, s_{k+1}^-, s_{k+2}, s_{k+3}, \dots), (s_0, \dots, s_{k-1}, s_k^+, s_{k+1}^+, s_{k+2}, s_{k+3}, \dots) \in \Sigma_0$. Let

$$\theta_k^+ = \kappa\big((s_0, \dots, s_{k-1}, s_k^+, s_{k+1}^+, s_{k+2}, s_{k+3}, \dots)\big),$$
$$\theta_k^- = \kappa\big((s_0, \dots, s_{k-1}, s_k^-, s_{k+1}^-, s_{k+2}, s_{k+3}, \dots)\big).$$

It is easy to check that $\theta_k^- < \theta < \theta_k^+$ and $\mathbf{J}(\theta_k^+, \theta) = \mathbf{J}(\theta_k^-, \theta) = k \to \infty$ as $k \to \infty$. $\square$

4.3. Modified puzzle piece

Consistent with the idea of the 'thickened puzzle piece' used in [21] to study the quadratic Julia set, we construct the 'modified puzzle piece' for McMullen maps. The 'modified puzzle piece' can be used to study the local connectivity of $J(f_\lambda)$ in the non-renormalizable case (see Lemma 7.1). It is also used to define renormalizations (see Remark 5.1).

Given an angle $\theta \in \Theta$ with itinerary $\mathbf{s}(\theta) = (s_0, s_1, s_2, \dots)$, the cut ray Ω_λ^θ is identified as $\Omega_\lambda^\theta = \bigcap_{k \geqslant 0} f_\lambda^{-k}(S_{s_k} \cup S_{-s_k})$; it can be approximated by the sequence of compact sets $\{\Omega_{\lambda,m}^\theta = \bigcap_{0 \leqslant k \leqslant m} f_\lambda^{-k}(S_{s_k} \cup S_{-s_k})\}_{m \geqslant 0}$ in Hausdorff topology. Now, we consider the set $\bar{\mathbb{C}} \setminus \Omega_{\lambda,m}^\theta$. The open set $\bar{\mathbb{C}} \setminus \Omega_{\lambda,m}^\theta$ consists of two connected components, and the boundary of each component is a Jordan curve. Denote these two boundary curves by $\gamma_{\lambda,m}^1(\theta)$ and $\gamma_{\lambda,m}^2(\theta)$. Let $V_m(\theta) = \gamma_{\lambda,m}^1(\theta) \cap \gamma_{\lambda,m}^2(\theta)$ be the intersection of these two curves. It is obvious that $V_m(\theta)$ consists of finitely many points and that $V_m(\theta) = \Omega_\lambda^\theta \cap (\bigcup_{0 \leqslant k \leqslant m+1} f_\lambda^{-k}(\infty))$. For any $v \in V_m(\theta)$, let $D(v)$ be the connected component of $\{z \in A_\lambda;\ G_\lambda(z) > 1\}$ that contains v. Obviously, $D(v)$ is a disk.

In the following, we construct the 'modified puzzle piece'. For the Yoccoz puzzle induced by the graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$, recall that each puzzle piece P_0 of depth zero is contained in a unique component of $\bar{\mathbb{C}} \setminus g_\lambda(\theta_1, \dots, \theta_N)$. This component is simply connected and is denoted by Q_0. We may choose a m large enough so that for any $\alpha, \beta \in \{\tau^k(\theta_j);\ 1 \leqslant j \leqslant N,\ k \geqslant 0\}$ with $\Omega_\lambda^\alpha \neq \Omega_\lambda^\beta$,

$$\Omega_{\lambda,m}^\alpha \cap \Omega_{\lambda,m}^\beta = \Omega_\lambda^\alpha \cap \Omega_\lambda^\beta.$$

The disk Q_0 is bounded by some collection of cut rays, say $\{\Omega_\lambda^\alpha; \alpha \in \Lambda(Q_0)\}$, where $\Lambda(Q_0)$ is an index set induced by Q_0. For any $\alpha \in \Lambda(Q_0)$, choose a curve $\gamma(\alpha) \in \{\gamma_{\lambda,m}^1(\alpha), \gamma_{\lambda,m}^2(\alpha)\}$ such that $\gamma(\alpha) \cap Q_0 = \emptyset$. Let $\hat{Q}_0$ be the connected component of $\bar{\mathbb{C}} \setminus \bigcup_{\alpha \in \Lambda(Q_0)} \gamma(\alpha)$ that contains Q_0, and let $V(Q_0) = \bigcup_{\alpha \in \Lambda(Q_0)} (V_m(\alpha) \cap \partial Q_0)$. The modified puzzle piece $\hat{P}_0$ of P_0 is defined as follows:

$$\hat{P}_0 = \hat{Q}_0 - \bigcup_{v \in V(Q_0)} \overline{D(v)}.$$

Roughly speaking, we can obtain $\hat{P}_0$ from Q_0 by thickening Q_0 near $\partial Q_0 \setminus V(Q_0)$ and truncating Q_0 near the points in $V(Q_0)$. The puzzle piece P_0 is not contained in $\hat{P}_0$; for this reason, we call $\hat{P}_0$ the 'modified puzzle piece' of P_0 rather than the 'thickened puzzle piece' of P_0.

Modified puzzle pieces of greater depth can be constructed by the usual inductive procedure; if $\hat{P}_d^{(j)}$ is the modified puzzle piece of depth d, then each component of $f_\lambda^{-1}(\hat{P}_d^{(j)})$ is the modified puzzle piece of depth $d+1$ (see Fig. 12).

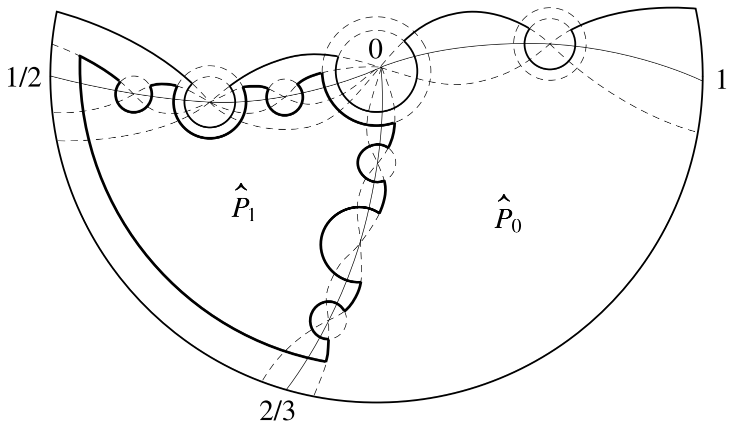

Fig. 12. An example of 'modified puzzle pieces', to depth one.

The advantage of these modified puzzle pieces is as follows: if a puzzle piece $P_d^{(j)}$ contains $P_{d+1}^{(k)}$, then the modified puzzle piece $\hat{P}_d^{(j)}$ contains $\overline{\hat{P}_{d+1}^{(k)}}$, which can be easily proved by induction. In other words, this construction replaces all of our annuli with non-degenerate annuli.

For $z \in \bar{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, let $\hat{P}_d(z)$ be the modified puzzle piece of $P_d(z)$. We will only make use of modified puzzle pieces that are small enough to satisfy the following additional restriction: if $\hat{P}_d(z)$ contains a critical point, then $P_d(z)$ must already contain this critical point. Note that if the graph $G_\lambda(\theta_1, \dots, \theta_N)$ is not touchable, then this requirement is easily satisfied for any bounded value of depth d by choosing m large enough, which will suffice for the applications.

Based on construction, the puzzle piece $P_d(z)$ and the modified puzzle piece $\hat{P}_d(z)$ satisfy the following relation:

$$\overline{P_d(z)} \subset \hat{P}_d(z) \cup A_\lambda, \qquad \bigcap_{d \geqslant 0} \overline{P_d(z)} \subset \bigcap_{d \geqslant 0} \hat{P}_d(z).$$

The modified puzzle pieces also satisfy the following symmetry properties: For any $z \in \bar{\mathbb{C}} \setminus (A_\lambda \cup J_0)$,

$$-\hat{P}_0(z) = \hat{P}_0(-z), \qquad \omega \hat{P}_d(z) = \hat{P}_d(\omega z), \qquad \omega^{2n} = 1, \quad d \geqslant 1.$$

4.4. Tableaux

In this section, we present some basic information on tableaux, based on Milnor's Lecture [21]. Applications of tableaux analysis combined with puzzle techniques can be found in [2,14, 21,22,25–27,32] and many other papers.

Recall that J_0 is the set of all points on $J(f_\lambda)$ with orbits that eventually touch the graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$. For $x \in \bar{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, the tableau $T(x)$ is defined as the two-dimensional array $(P_{d,l}(x))_{d,l \geqslant 0}$, where $P_{d,l}(x) = f_\lambda^l(P_{d+l}(x)) = P_d(f_\lambda^l(x))$. The position (d, l) is called critical if $P_{d,l}(x)$ contains a critical point in C_λ. If $P_{d,l}(x)$ contains a critical point $c \in C_\lambda$, the position (d, l) is called a c-position.

For any $x \in \bar{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, the tableau $T(x)$ satisfies the following three rules:

(T1) For each column $l \geqslant 0$, either the position (d,l) is critical for all $d \geqslant 0$ or there is a unique integer $d_0 \geqslant 0$ such that the position (d,l) is critical for all $d < d_0$ and not critical for $d \geqslant d_0$.

(T2) If $P_{d,l}(x) = P_d(y)$ for some $y \in \bar{\mathbb{C}} \setminus (A_\lambda \cup J_0)$, then $P_{i,l+j}(x) = P_{i,j}(y)$ for $0 \leqslant i+j \leqslant d$.

(T3) Let $T(c)$ be a tableau with $c \in C_\lambda$. Assume

(a) $P_{d+1-l,l}(c) = P_{d+1-l}(c')$ for some critical point $c' \in C_\lambda$, $0 \leqslant l < d$, and $P_{d-i,i}(c)$ contains no critical points for $0 < i < l$.

(b) $P_{d,m}(x) = P_d(c)$ and $P_{d+1,m}(x) \neq P_{d+1}(c)$ for some $m > 0$.

Then, $P_{d+1-l,m+l}(x) \neq P_{d+1-l}(c')$.

Remark 4.2. The tableau rule (T3) is based on the fact that every puzzle piece of depth $d \geqslant 1$ contains at most one critical point in C_λ.

Definition 4.2. 1. The tableau $T(x)$ is non-critical if there is an integer $d_0 \geqslant 0$ such that (d_0, j) is not critical for all $j > 0$. Otherwise, $T(x)$ is called critical. (One should be careful to note that $T(x)$ is critical does not mean $x \in C_\lambda$.)

2. The tableau $T(x)$ is called pre-periodic if there exist two integers $l \geqslant 0$ and $p \geqslant 1$ such that $P_{d,l+p}(x) = P_{d,l}(x)$ for all $d \geqslant 0$. In this case, if $l = 0$, $T(x)$ is called periodic, and the smallest integer $p \geqslant 1$ is called the period of $T(x)$.

3. Let $\mathrm{Row}_c(d)$ be the d-th row of the tableau $T(c)$ with $c \in C_\lambda$. We say $\mathrm{Row}_c(d+l)$ with $l > 0$ is a child of $\mathrm{Row}_c(d)$ if there is a critical point $c' \in C_\lambda$ such that $A_d(f_\lambda^l(c)) = A_d(c')$ and $f_\lambda^l : A_{d+l}(c) \to A_d(c')$ is a degree two covering map.

4. Let $c \in C_\lambda$. For $d \geqslant 1$, we say $\mathrm{Row}_c(d)$ is excellent if $A_d(f_\lambda^l(c))$ is not semi-critical for all $l \geqslant 0$.

Remark 4.3. By Lemma 4.1 and the fact that $f_\lambda^k(\omega z) = \pm f_\lambda^k(z)$ for $k \geqslant 1$, $\omega^{2n} = 1$, we have

1. If (d,l) is a critical position for some tableau $T(c)$ with $c \in C_\lambda$, then (d,l) is a critical position of $T(c')$ for every $c' \in C_\lambda$.

2. If there is $c \in C_\lambda$ such that the tableau $T(c)$ is critical, non-critical or pre-periodic, then for every $c' \in C_\lambda$, the tableau $T(c')$ is critical, non-critical or pre-periodic, respectively.

3. If $\mathrm{Row}_c(d)$ is excellent or has a child $\mathrm{Row}_c(d+l)$ for some critical point $c \in C_\lambda$, then for every $c' \in C_\lambda$, $\mathrm{Row}_{c'}(d)$ is excellent or has a child $\mathrm{Row}_{c'}(d+l)$, respectively.

Lemma 4.7. *Suppose some tableau $T(c)$ with $c \in C_\lambda$ is critical but not pre-periodic, then*

1. *For every $d \geqslant 1$,* $\mathrm{Row}_c(d)$ *has at least one child.*
2. *If* $\mathrm{Row}_c(d)$ *is excellent, then* $\mathrm{Row}_c(d)$ *has at least two children.*
3. *If* $\mathrm{Row}_c(d)$ *is excellent and* $\mathrm{Row}_c(d+l)$ *is its child, then* $\mathrm{Row}_c(d+l)$ *is also excellent.*
4. *If* $\mathrm{Row}_c(d)$ *has only one child, say* $\mathrm{Row}_c(d+l)$*, then* $\mathrm{Row}_c(d+l)$ *is excellent.*

Proof. 1. By hypothesis, for every $d \geqslant 1$, we can find a smallest integer $l > 0$ such that the annulus $A_d(f_\lambda^l(c))$ is c'-critical for some $c' \in C_\lambda$. The map $f_\lambda^l : A_{d+l}(c) \to A_d(c')$ is a degree two covering map, which implies that $\mathrm{Row}_c(d+l)$ is a child of $\mathrm{Row}_c(d)$.

2. Following 1, there exists $d' > d$ such that the annulus $A_{d'}(f_\lambda^l(c))$ is c'-semi-critical. Because $\mathrm{Row}_c(d)$ is excellent, by tableau rule (T3), $A_{d'-t}(f_\lambda^{l+t}(c))$ is either off-critical or semi-critical for $0 < t \leqslant d' - d$. In particular, $A_d(f_\lambda^{l+d'-d}(c))$ is off-critical. Hence, we can find a

smallest integer $l' > l + d' - d$ such that the annulus $A_d(f_\lambda^{l'}(c))$ is critical; therefore, $\mathrm{Row}_c(d+l')$ is another child of $\mathrm{Row}_c(d)$.

3. If $\mathrm{Row}_c(d+l)$ is not excellent, then there is a column $l' \geqslant l$ such that $A_{d+l}(f_\lambda^{l'}(c))$ is semi-critical. By tableau rule (T3), $A_d(f_\lambda^{l+l'}(c))$ is also semi-critical, which contradicts the fact that $\mathrm{Row}_c(d)$ is excellent.

4. If $\mathrm{Row}_c(d+l)$ is not excellent, then as in (3), $A_d(f_\lambda^{l+l'}(c))$ is semi-critical for some $l' \geqslant l$. Suppose $l' \geqslant l$ is the smallest integer. We can find a smallest integer $t > l' + l$ such that $A_d(f_\lambda^t(c))$ is c'-critical for some $c' \in C_\lambda$. Then $\mathrm{Row}_c(d+t)$ is also a child of $\mathrm{Row}_c(d)$, which is a contradiction. $\square$

Lemma 4.8. *Suppose some tableau $T(c)$ with $c \in C_\lambda$ is critical and pre-periodic.*

1. *If n is odd, then there exist exactly two critical points $\pm c' \in C_\lambda$ such that $T(c')$ and $T(-c')$ are periodic.*
2. *If n is even, then there is a unique critical point $\tilde{c} \in C_\lambda$ such that $T(\tilde{c})$ is periodic.*

Proof. Because $T(c)$ is critical and pre-periodic, there exist a smallest integer $p \geqslant 1$ and a unique critical point $c' \in C_\lambda$ such that (d, p) is a c'-position for all $d \geqslant 0$.

1. If n is odd, there are two possibilities: either $f_\lambda(c) = f_\lambda(c')$ or $f_\lambda(c) + f_\lambda(c') = 0$.

If $f_\lambda(c) = f_\lambda(c')$, then both $T(c')$ and $T(-c')$ are periodic with period p. In this case, there is an integer $d_0 \geqslant 0$ such that for any $d \geqslant d_0$, $0 < l < p$, the position (d, l) is not critical. It is easy to check that for any $\tilde{c} \in C_\lambda \setminus \{\pm c'\}$, the tableau $T(\tilde{c})$ is strictly pre-periodic. In particular, if $p = 1$, then $P_d(c') = P_d(f_\lambda(c'))$ for all $d \geqslant 0$. This means that for any $d \geqslant 0$, c' and $f_\lambda(c')$ lie in the same puzzle piece of depth d. Thus, we conclude $\{\pm c'\} = \{c_0, c_n\}$.

If $f_\lambda(c) + f_\lambda(c') = 0$, then both $T(c')$ and $T(-c')$ are periodic with period $2p$. Consider the tableau $T(c')$; there is an integer $d_0 \geqslant 0$ such that for any $d \geqslant d_0$, $0 < l < p$, the position (d, l) is not critical and for any $d \geqslant 0$ the position (d, p) is $(-c')$-critical. It is easy to confirm that for any $\tilde{c} \in C_\lambda \setminus \{\pm c'\}$, the tableau $T(\tilde{c})$ is strictly pre-periodic. In particular, if $p = 1$, then $P_d(-c') = P_d(f_\lambda(c'))$ for all $d \geqslant 0$. Therefore, for any $d \geqslant 0$, $-c'$ and $f_\lambda(c')$ lie in the same puzzle piece of depth d. Thus, we conclude $\{\pm c'\} = \{c_1, c_{n+1}\}$.

2. n is even. In this case, based on the fact that $f_\lambda^k(v_\lambda^+) = f_\lambda^k(v_\lambda^-)$ for all $k \geqslant 1$, we conclude the tableau $T(f_\lambda(c'))$ is periodic With a period p and the tableau $T(-f_\lambda(c'))$ is strictly pre-periodic. There is thus a unique critical point $\tilde{c} \in f_\lambda^{-1}(f_\lambda(c'))$ such that $T(\tilde{c})$ is periodic. For this tableau, there is an integer $d_0 \geqslant 0$ such that for any $d \geqslant d_0$, $0 < l < p$, the position (d, l) is not critical. It is easy to check that for any $c'' \in C_\lambda \setminus \{\tilde{c}\}$, the tableau $T(c'')$ is strictly pre-periodic. In particular, if $p = 1$ and $T(v_\lambda^+)$ is periodic, then $\tilde{c} = c_0$; if $p = 1$ and $T(v_\lambda^-)$ is periodic, then $\tilde{c} = c_{n+1}$. $\square$

5. Renormalizations

In this section, we discuss the renormalization of McMullen maps with respect to the puzzle piece.

Definition 5.1. If there exist a critical point c of f_λ, an integer $p \geqslant 1$ and two disks U and V containing c such that

$$\epsilon f_\lambda^p : U \to V$$

is a quadratic-like map whose Julia set is connected (here $\epsilon \in \{\pm 1\}$ is a symbol), then we say f_λ is p-renormalizable at c if $\epsilon = 1$ and f_λ is p-$*$-renormalizable at c if $\epsilon = -1$. In the former case, the triple (f_λ^p, U, V) is called a p-renormalization of f_λ at c. In the latter case, the triple $(-f_\lambda^p, U, V)$ is called a p-$*$-renormalization of f_λ at c.

In the following, we use $K_c = \{z \in U;\ (\epsilon f_\lambda^p)^k(z) \in U,\ \forall k \geqslant 0\} = \bigcap_{k\geqslant 0}(\epsilon f_\lambda^p)^{-k}(U)$ to denote the small filled Julia set of the ($*$-)renormalization $(\epsilon f_\lambda^p, U, V)$. By the straightening theorem of Douady and Hubbard [11], if $(\epsilon f_\lambda^p, U, V)$ is a p-($*$-)renormalization of f_λ, then ϵf_λ^p is conjugated by a quasi-conformal map σ to a unique quadratic polynomial $p_\mu(z) = z^2 + \mu$ in a neighborhood of the filled Julia set K_c. Let β be the β-fixed point (i.e., the landing point of the zero external ray) of p_μ and β' be the other preimage of β. We call $\beta_c = \sigma^{-1}(\beta)$ the β-fixed point of the renormalization $(\epsilon f_\lambda^p, U, V)$. The other preimage of β_c under the renormalization is $\beta_c' = \sigma^{-1}(\beta')$.

In this section, we always assume that the graph $\mathbf{G}_\lambda(\theta_1, \ldots, \theta_N)$ is admissible.

5.1. From tableau to renormalizations

Lemma 5.1. *Suppose some tableau $T(c)$ with $c \in C_\lambda$ is pre-periodic.*

1. *If $T(c)$ is non-critical, then f_λ is critically finite.*
2. *If $T(c)$ is critical, then f_λ is either renormalizable or $*$-renormalizable.*

Proof. Because $T(c)$ is pre-periodic, there exist two integers $l \geqslant 0$ and $p \geqslant 1$ such that $P_d(f_\lambda^{l+p}(c)) = P_{d,l+p}(c) = P_{d,l}(c) = P_d(f_\lambda^l(c))$ for all $d \geqslant 0$.

1. $T(c)$ is non-critical. In this case, the tableaux $T(f_\lambda^l(c))$ and $T(f_\lambda^{l+p}(c))$ are also non-critical. Based on Lemma 7.1, $\{f_\lambda^{l+p}(c)\} = \bigcap_{d\geqslant 0} P_d(f_\lambda^{l+p}(c)) = \bigcap_{d\geqslant 0} P_d(f_\lambda^l(c)) = \{f_\lambda^l(c)\}$. Therefore, $f_\lambda^{l+p}(c) = f_\lambda^l(c)$, and f_λ is critically finite.

2. $T(c)$ is critical. If n is odd, then based on Lemma 4.8, there are exactly two critical points $\pm c' \in C_\lambda$ such that $T(c')$ and $T(-c')$ are periodic. Suppose the period is p, and consider the tableau $T(c')$. There are two possibilities:

Case 1. There is an integer $d_0 \geqslant 0$ such that for any $d \geqslant d_0$, $0 < l < p$, the position (d, l) is not critical. Then, $f_\lambda^p : P_{d_0+p}(c') \to P_{d_0}(c')$ is a quadratic-like map and $\{f_\lambda^{kp}(c');\ k \geqslant 0\} \subset P_{d_0+p}(c')$. Thus, $(f_\lambda^p, P_{d_0+p}(c'), P_{d_0}(c'))$ is a p-renormalization of f_λ at c'. Because f_λ is an odd function, $(f_\lambda^p, P_{d_0+p}(-c'), P_{d_0}(-c'))$ is a p-renormalization of f_λ at $-c'$.

Case 2. p is even and there is an integer $d_0 \geqslant 0$ such that for any $d \geqslant d_0$, $0 < l < p/2$, the position (d, l) is not critical, and for any $d \geqslant 0$, the position $(d, p/2)$ is $(-c')$-critical. Then, $-f_\lambda^{p/2} : P_{d_0+p/2}(c') \to P_{d_0}(c')$ is a quadratic-like map with $\{(-1)^k f_\lambda^{kp/2}(c');\ k \geqslant 0\} \subset P_{d_0+p/2}(c')$. Thus, $(-f_\lambda^{p/2}, P_{d_0+p/2}(c'), P_{d_0}(c'))$ is a $p/2$-$*$-renormalization of f_λ at c'. It turns out that $(-f_\lambda^{p/2}, P_{d_0+p/2}(-c'), P_{d_0}(-c'))$ is a $p/2$-$*$-renormalization of f_λ at $-c'$.

If n is even, then based on Lemma 4.8, there is a unique critical point $\tilde{c} \in C_\lambda$ such that $T(\tilde{c})$ is periodic. Suppose the period is p; there is then an integer $d_0 \geqslant 0$ such that for any $d \geqslant d_0$, $0 < l < p$, the position (d, l) is not critical. Then, $f_\lambda^p : P_{d_0+p}(\tilde{c}) \to P_{d_0}(\tilde{c})$ is a quadratic-like map and $\{f_\lambda^{kp}(\tilde{c});\ k \geqslant 0\} \subset P_{d_0+p}(\tilde{c})$. Thus, $(f_\lambda^p, P_{d_0+p}(\tilde{c}), P_{d_0}(\tilde{c}))$ is a p-renormalization of f_λ at $\tilde{c}$. Because f_λ is an even function, $(-f_\lambda^p, P_{d_0+p}(-\tilde{c}), P_{d_0}(-\tilde{c}))$ is a p-$*$-renormalization of f_λ at $-\tilde{c}$. $\square$

Remark 5.1. Lemma 5.1 also holds when the graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$ is not touchable. Indeed, in this case, we can use modified puzzle pieces to define renormalizations.

Proposition 5.1. *Suppose f_λ has a non-repelling cycle in $\mathbb{C}$; then f_λ is either renormalizable or $*$-renormalizable. In this situation, there are three possibilities:*

1. *If f_λ is renormalizable and n is odd, then f_λ has exactly two non-repelling cycles in $\mathbb{C}$.*
2. *If f_λ is $*$-renormalizable and n is odd, then f_λ has exactly one non-repelling cycle in $\mathbb{C}$.*
3. *If f_λ is renormalizable and n is even, then f_λ has exactly one non-repelling cycle in $\mathbb{C}$.*

Proof. Let $\mathcal{C} = \{z_0, f_\lambda(z_0), \dots, f_\lambda^q(z_0) = z_0\}$ be the non-repelling cycle of f_λ in $\mathbb{C}$. By Proposition 4.1, we can find an admissible graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$. By Proposition 3.4, the cycle $\mathcal{C}$ avoids the graph $\mathbf{G}_\lambda(\theta_1, \dots, \theta_N)$. Thus, for any $z \in \mathcal{C}$ and any integer $d \geqslant 0$, the puzzle piece $P_d(z)$ is well defined.

We claim that there exist $z \in \mathcal{C}$ and a critical point $c \in C_\lambda$ such that $P_d(z) = P_d(c)$ for all $d \geqslant 0$. Otherwise, the tableau $T(z)$ is non-critical for any $z \in \mathcal{C}$. It follows that there is an integer $d_0 \geqslant 0$ such that the map $f_\lambda^q : P_{d_0+q}(z_0) \to P_{d_0}(z_0)$ is conformal. Based on the Schwarz Lemma, $|(f_\lambda^q)'(z_0)| > 1$, which is a contradiction.

In this way, we can find a critical point $c \in C_\lambda$ with tableau $T(c)$ that is periodic. Based on Lemma 5.1, f_λ is either renormalizable or $*$-renormalizable.

To continue, suppose the period of $T(c)$ is p, which is necessarily a divisor of q. Based on Lemma 5.1, there are three possibilities:

(P1). n is odd and $(f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is a p-renormalization of f_λ at c. In this case, $(f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is quasi-conformally conjugate to a polynomial $z \mapsto z^2 + \mu$. Because a quadratic polynomial has at most one non-repelling cycle (see [3] or [28]), it turns out that $\mathcal{C}$ is the only non-repelling cycle contained in $\bigcup_{0 \leqslant j < p} f_\lambda^j(K_c)$. On the other hand, $-\mathcal{C}$ is the only non-repelling cycle contained in $\bigcup_{0 \leqslant j < p} f_\lambda^j(-K_c)$. Because there are exactly two critical points whose tableaux are periodic in this case and $(\bigcup_{0 \leqslant j < p} f_\lambda^j(K_c)) \cap (\bigcup_{0 \leqslant j < p} f_\lambda^j(-K_c)) = \emptyset$, we conclude that f_λ has exactly two non-repelling cycles in $\mathbb{C}$.

(P2). n is odd and $(-f_\lambda^{p/2}, P_{d_0+p/2}(c), P_{d_0}(c))$ is a $p/2$-$*$-reorganization of f_λ at c. In this case, the cycle $\mathcal{C}$ meets both K_c and $-K_c$. By a similar argument as above, one sees that $\mathcal{C}$ is the only non-repelling cycle contained in $\bigcup_{0 \leqslant j < p} f_\lambda^j(K_c)$. Because the cycle $-\mathcal{C}$ is also contained in $\bigcup_{0 \leqslant j < p} f_\lambda^j(K_c)$, it turns out that $\mathcal{C} = -\mathcal{C}$.

(P3). n is even and $(f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is a p-renormalization of f_λ at c. In this case, c is the only critical point whose tableau $T(c)$ is periodic. Based on a similar argument as made above, we show that $\mathcal{C}$ is the only non-repelling cycle in $\mathbb{C}$. $\square$

In the following, we discuss the case when f_λ has an indifferent cycle of multiplier $e^{2\pi i\theta}$. Douady [9] conjectured that for any rational map, whenever it is linearizable (i.e., the map is conformally conjugate to an irrational rotation) near an indifferent fixed point of multiplier $e^{2\pi i\theta}$, then θ must be a Brjuno number. Here, an irrational number θ of convergents p_k/q_k (rational approximations obtained by the continued fraction expansion) is a Brjuno number (denoted by $\mathcal{B}$) if

$$\sum_{k\geqslant 1}\frac{\log q_{k+1}}{q_k}<+\infty.$$

According to Cremer, Siegel and Brjuno, if $\theta\in\mathcal{B}$, then every germ $f(z)=e^{2\pi i\theta}z+\mathcal{O}(z^2)$ is linearizable. Yoccoz [33] shows that if the quadratic polynomial $z\mapsto e^{2\pi i\theta}z+z^2$ is linearizable, then $\theta\in\mathcal{B}$. For a general case, Geyer [12] shows that for any $d\geqslant 2$, if $z\mapsto z^d+c$ has an indifferent cycle of multiplier $e^{2\pi i\theta}$ near which the map is linearizable, then $\theta\in\mathcal{B}$. Based on these results and Proposition 5.1, we immediately establish:

Proposition 5.2. *Suppose f_λ has an indifferent cycle of multiplier $e^{2\pi i\theta}$; then f_λ is linearizable near the indifferent cycle if and only if $\theta\in\mathcal{B}$.*

5.2. Properties of renormalizations

In this section, we assume that some tableau $T(c)$ with $c\in C_\lambda$ is periodic with period k. By Lemma 5.1, f_λ is either k-renormalizable at c or $k/2$-$*$-renormalizable at c. Let $(\epsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ be the corresponding renormalization, where

$$(\epsilon,p)=\begin{cases}(1,k), & \text{if } f_\lambda \text{ is } k\text{-renormalizable at } c,\\ (-1,k/2), & \text{if } f_\lambda \text{ is } k/2\text{-}*\text{-renormalizable at } c.\end{cases}$$

The small filled Julia set $K_c=\bigcap_{d\geqslant 0}\overline{P_d(c)}=\bigcap_{d\geqslant 0}P_d(c)$.

If $K_c\cap\partial B_\lambda\neq\emptyset$, we will show that there is a unique external ray in B_λ converging on K_c. Before the proof, we need a classic result for quadratic polynomials:

Lemma 5.2. *Let $p_\mu(z)=z^2+\mu$ be a quadratic polynomial with a connected filled Julia set K. If there is a curve $\delta\subset\mathbb{C}\setminus K$ converging to $x\in K$ and $p_\mu(\delta)\supset\delta$, then x is the β-fixed point of p_μ.*

Here, a curve $\delta\subset\mathbb{C}\setminus K$ converges to $x\in K$ means that δ can be parameterized as $\delta:[0,1)\to\mathbb{C}\setminus K$ such that $\lim_{t\to 1}\delta(t)$ exists and $\lim_{t\to 1}\delta(t)=x\in K$. See [19] for a proof of Lemma 5.2. The conclusion also holds for quadratic-like maps.

Lemma 5.3. *Suppose some tableau $T(c)$ with $c\in C_\lambda$ is k-periodic and $K_c\cap\partial B_\lambda\neq\emptyset$, then*

1. *The small filled Julia sets $K_c, f_\lambda(K_c),\ldots,f_\lambda^{k-1}(K_c)$ are pairwise disjoint.*
2. *There is a unique external ray $R_\lambda(t)$ in B_λ accumulating on K_c. This external ray lands at $\beta_c\in K_c$ and the angle t is k-periodic.*

Proof. 1. If $f_\lambda^i(K_c)\cap f_\lambda^j(K_c)\neq\emptyset$ for some $0\leqslant i<j<k$, then $K_c\cap f_\lambda^{k+i-j}(K_c)\neq\emptyset$. Thus, $P_{d,k+i-j}(c)=f_\lambda^{k+i-j}(P_{d+k+i-j}(c))=P_d(c)$ for all $d\geqslant 0$. This implies that the tableau $T(c)$ is $(k+i-j)$-periodic, which is a contradiction.

2. First, note that $f_\lambda^k(P_{d+k}(c)) = P_d(c)$ for $d \geqslant 0$. Because $K_c \cap \partial B_\lambda \neq \emptyset$, $P_{mk}(c) \cap B_\lambda$ is nonempty and bounded by two external rays, say $R_\lambda(\theta_m^-)$ and $R_\lambda(\theta_m^+)$ with $\theta_m^- < \theta_m^+$. Let $Q(\theta_m^-, \theta_m^+) = \overline{P_{mk}(c) \cap B_\lambda}$, $m \geqslant 1$. Because $f_\lambda^k(Q(\theta_{m+1}^-, \theta_{m+1}^+)) = Q(\theta_m^-, \theta_m^+)$, we have

$$\theta_m^- \leqslant \theta_{m+1}^- \leqslant \cdots \leqslant \theta_{m+1}^+ \leqslant \theta_m^+, \qquad \theta_m^+ - \theta_m^- = n^k\big(\theta_{m+1}^+ - \theta_{m+1}^-\big).$$

Thus, there is a common limit $t = \lim \theta_m^+ = \lim \theta_m^-$. Because $\theta_m^- \leqslant t \leqslant \theta_m^+$ for any m, we have $n^k t \equiv t \pmod{\mathbb{Z}}$. Thus, t is a periodic angle and the external ray $R_\lambda(t)$ lands at a point $z \in K_c \cap \partial B_\lambda$ (because rational external rays always land). Because $R_\lambda(n^j t)$ lands at $f_\lambda^j(z) \in f_\lambda^j(K_c) \cap \partial B_\lambda$ for $0 \leqslant j < k$ and the small filled Julia sets $K_c, f_\lambda(K_c), \ldots, f_\lambda^{k-1}(K_c)$ are pairwise disjoint, we conclude that the angles $t, nt, \ldots, n^{k-1}t$ are distinct. Thus, t is k-periodic.

Suppose θ is another angle such that the external ray $R_\lambda(\theta)$ accumulates on K_c. Then, $\theta_m^- \leqslant \theta \leqslant \theta_m^+$ for any m. Thus, $\theta = \lim \theta_m^+ = \lim \theta_m^- = t$.

To finish, we show $z = \beta_c$. Because $T(c)$ is k-periodic, f_λ is either k-renormalizable or $k/2$-$*$-renormalizable. In the former case, $f_\lambda^k(R_\lambda(t)) = R_\lambda(t)$. Thus, based on Lemma 5.2, $z = \beta_c$. In the latter case, because $R_\lambda(t)$ is the unique external ray accumulating on K_c, we conclude that $R_\lambda(t + 1/2) = -R_\lambda(t)$ is the unique external ray accumulating on $-K_c$. On the other hand, $f_\lambda^{k/2}(R_\lambda(t))$ is also an external ray accumulating on $-K_c$, and we have $f_\lambda^{k/2}(R_\lambda(t)) = R_\lambda(t + 1/2) = -R_\lambda(t)$. In this case, $-f_\lambda^{k/2}(R_\lambda(t)) = R_\lambda(t)$. Again, based on Lemma 5.2, $z = \beta_c$. $\square$

6. A criterion of local connectivity

In this section, we present a criterion for the characterization of the local connectivity of the immediate basin of attraction. This criterion can be applied together with Yoccoz puzzle techniques to study the local connectivity and higher regularity of the boundary ∂B_λ.

In the following discussion, let f be a rational map of degree at least two, $C(f)$ be the critical set of f and $P(f) = \overline{\bigcup_{k \geqslant 1} f^k(C(f))}$ be the post-critical set. Suppose that f has an attracting periodic point z_0 and the immediate basin B of z_0 is simply connected. Let $B(z, \delta) = \{x \in \mathbb{C};\ |x - z| < \delta\}$.

Definition 6.1. We say f satisfies the **BD** (bounded degree) condition on ∂B if for any $u \in \partial B$ there is a number $\varepsilon_u > 0$ such that for any integer $m \geqslant 0$ and any component $U_m(u)$ of $f^{-m}(B(u, \varepsilon_u))$ intersecting with ∂B, $U_m(u)$ is simply connected and the degree $\deg(f^m : U_m(u) \to B(u, \varepsilon_u))$ is bounded by some constant D that is independent of u, m and $U_m(u)$.

The following is a remark on the definition: because $f^m : U_m(u) \to B(u, \varepsilon_u)$ is a proper map between two disks, we conclude by the Maximum Principle that for any disk $W \subset B(u, \varepsilon_u)$ and any component V of $f^{-m}(W)$ that lies inside $U_m(u)$, V is also a disk.

The aim of this section is to prove the following:

Proposition 6.1. *If f satisfies the **BD** condition on ∂B, then*

1. *∂B is locally connected.*
2. *If, furthermore, ∂B is a Jordan curve, then ∂B is a quasi-circle.*

Before presenting the proof, we introduce a distortion lemma. Let U be a hyperbolic disk in $\mathbb{C}$ and $z \in U$. The shape of U about z is defined by:

$$\operatorname{Shape}(U, z) = \sup_{x \in \partial U} |x - z| / \inf_{x \in \partial U} |x - z|.$$

It is obvious that $\operatorname{Shape}(U, z) = \infty$ if and only if U is unbounded and $\operatorname{Shape}(U, z) = 1$ if and only if U is a round disk centered at z. In all other cases, $1 < \operatorname{Shape}(U, z) < \infty$.

Let K be a connected and compact subset of U containing at least two points. For any $z_1, z_2 \in K$, define the turning of K about z_1 and z_2 by:

$$\Delta(K; z_1, z_2) = \operatorname{diam}(K)/|z_1 - z_2|,$$

where $\operatorname{diam}(\cdot)$ is the Euclidean diameter. It is obvious that $1 \leqslant \Delta(K; z_1, z_2) \leqslant \infty$ and $\Delta(K; z_1, z_2) = \infty$ if and only if $z_1 = z_2$.

Lemma 6.1. *For $i \in \{1, 2\}$, let (V_i, U_i) be a pair of hyperbolic disks in $\mathbb{C}$ with $\overline{U_i} \subset V_i$. $g : V_1 \to V_2$ is a proper holomorphic map of degree d, and U_1 is a component of $g^{-1}(U_2)$. Suppose $\operatorname{mod}(V_2 \setminus \overline{U_2}) \geqslant m > 0$. Then,*

1. *(Shape distortion) There is a constant $C(d, m) > 0$ such that for all $z \in U_1$,*

$$\operatorname{Shape}(U_1, z) \leqslant C(d, m) \operatorname{Shape}\bigl(U_2, g(z)\bigr).$$

2. *(Turning distortion) There is a constant $D(d, m) > 0$ such that for any connected and compact subset K of U_1 with $\#K \geqslant 2$ and any $z_1, z_2 \in K$,*

$$\Delta(K; z_1, z_2) \leqslant D(d, m) \Delta\bigl(g(K); g(z_1), g(z_2)\bigr).$$

Proof. A complete proof of 1 can be found in [30], Theorem 2.3.2. In the following, we prove 2. We assume that $g(z_1) \neq g(z_2)$. Otherwise, $\Delta(g(K); g(z_1), g(z_2)) = \infty$, and the conclusion follows. Let $\rho(x, y)$ be the hyperbolic distance in V_2, and let B_1, B_2 be two hyperbolic disks both centered at $g(z_1)$, with radii $\max_{\zeta \in g(K)} \rho(g(z_1), \zeta)$ and $\rho(g(z_1), g(z_2))$, respectively. Let $\varphi : V_2 \to D$ be the Riemann mapping with $\varphi(g(z_1)) = 0$, and let $W = \varphi(U_2)$. Because $\operatorname{mod}(\mathbb{D} \setminus \overline{W}) = \operatorname{mod}(V_2 \setminus \overline{U_2}) \geqslant m$, we conclude by the Grötzsch Theorem that there is a constant $r(m) \in (0, 1)$ such that $W \subset D_{r(m)}$; here, we use D_r to denote the disk $\{z;\ |z| < r\}$.

Note that $\varphi(B_1), \varphi(B_2)$ are two round disks, say D_R and D_r, centered at 0. Based on Koebe distortion, there exist three constants $C_1(m), C_2(m), C_3(m) > 0$ such that

$$\operatorname{Shape}\bigl(B_1, g(z_1)\bigr) \leqslant C_1(m), \qquad \operatorname{Shape}\bigl(B_2, g(z_1)\bigr) \leqslant C_2(m),$$
$$R/r \leqslant C_3(m) \max_{\zeta \in g(K) \cap \partial B_1} \bigl|g(z_1) - \zeta\bigr| / \bigl|g(z_1) - g(z_2)\bigr| \leqslant C_3(m) \Delta\bigl(g(K); g(z_1), g(z_2)\bigr).$$

For $i \in \{1, 2\}$, let W_i be the component of $g^{-1}(B_i)$ that contains z_1. Based on the Maximum Principle, W_1 and W_2 are simply connected. We may assume that $K \subset \overline{W}_1$ (otherwise, we can replace B_1 by $\hat{B}_1$, a hyperbolic disk centered at $g(z_1)$ with radius $\epsilon + \max_{\zeta \in g(K)} \rho(g(z_1), \zeta)$, where ϵ is a small positive constant and then let $\epsilon \to 0^+$). Thus, $\operatorname{diam}(K) \leqslant \operatorname{diam}(W_1) \leqslant$

$2\sup_{\zeta\in\partial W_1}|\zeta - z_1|$. Consider the location of z_2, which by the Maximum Principle is either $z_2 \in \partial W_2$ or $z_2 \in U_1 \setminus \overline{W}_2$. In either case, $|z_1 - z_2| \geqslant \inf_{\zeta\in\partial W_2}|\zeta - z_1|$. Thus, by Shape distortion,

$$\begin{aligned}\Delta(K; z_1, z_2) &\leqslant 2\sup_{\zeta\in\partial W_1}|\zeta - z_1|/\inf_{\zeta\in\partial W_2}|\zeta - z_1| \\ &= 2\mathrm{Shape}(W_1, z_1)\mathrm{Shape}(W_2, z_1)Q(W_1, W_2, z_1) \\ &\leqslant C_1(d, m)\mathrm{Shape}\big(B_1, g(z_1)\big)\mathrm{Shape}\big(B_2, g(z_1)\big)Q(W_1, W_2, z_1) \\ &\leqslant C_2(d, m)Q(W_1, W_2, z_1)\end{aligned}$$

where $Q(W_1, W_2, z_1) = \inf_{\zeta\in\partial W_1}|\zeta - z_1|/\sup_{\zeta\in\partial W_2}|\zeta - z_1|$. To finish, in the following we show that there is a constant $c(m) > 0$ such that

$$Q(W_1, W_2, z_1) \leqslant c(m)\Delta\big(g(K); g(z_1), g(z_2)\big).$$

In fact, we only need to consider the case $Q(W_1, W_2, z_1) > 1$. In this case, the annulus $W_1 \setminus \overline{W}_2$ contains the round annulus $\{w \in \mathbb{C};\ \sup_{\zeta\in\partial W_2}|\zeta - z_1| < |w - z_1| < \inf_{\zeta\in\partial W_1}|\zeta - z_1|\}$. It turns out that

$$\begin{aligned}\frac{1}{2\pi}\log Q(W_1, W_2, z_1) \leqslant \mathrm{mod}(W_1 \setminus \overline{W}_2) \leqslant \mathrm{mod}(B_1 \setminus \overline{B}_2) = \frac{1}{2\pi}\log\frac{R}{r} \\ \leqslant \frac{1}{2\pi}\log\big(C_3(m)\Delta\big(g(K); g(z_1), g(z_2)\big)\big).\end{aligned}$$

The conclusion follows. □

Proof of Proposition 6.1. By replacing f with f^k, we assume z_0 is a fixed point of f. Based on quasi-conformal surgery, we assume z_0 is a superattracting fixed point with local degree $d = \deg(f : B \to B) \geqslant 2$. Thus, B contains no critical points other than z_0. By Möbius conjugation, we assume $z_0 = \infty$.

Because f satisfies the **BD** condition on ∂B, there exists a constant $\delta > 0$ such that for any $u \in \partial B$, any integer $m \geqslant 0$ and any component $U_m(u)$ of $f^{-m}(B(u, \delta))$ that intersects with ∂B, $U_m(u)$ is simply connected and $\deg(f^m : U_m(u) \to B(u, \delta)) \leqslant D$. In fact, we can choose δ as the Lebesgue number of the family $\mathcal{F} = \{B(u, \varepsilon_u);\ u \in \partial B\}$, which is an open covering of the boundary ∂B.

The proof consists of four steps, as follows:

Step 1. *Let $V_m(z)$ be the component of $f^{-m}(B(z, \delta/2))$ contained in $U_m(z)$ and intersecting with ∂B, then*

$$\lim_{m\to\infty}\sup_{z\in\partial B}\mathrm{diam}\big(V_m(z)\big) = 0.$$

Otherwise, there is a constant $d_0 \geqslant 0$ and two sequences $\{z_k\} \subset \partial B$ and $\{\ell_k\}$ such that $\mathrm{diam}(V_{\ell_k}(z_k)) \geqslant d_0$. For every $k \geqslant 1$, choose a point $y_k \in f^{-\ell_k}(z_k) \cap V_{\ell_k}(z_k)$. By passing to a subsequence, we assume $y_k \to y_\infty \in \partial B$ and $z_k \to z_\infty \in \partial B$. Based on Lemma 6.1, there is a constant $C(D)$ such that

$$\mathrm{Shape}\big(V_{\ell_k}(z_k), y_k\big) \leqslant C(D)\mathrm{Shape}\big(B(z_k, \delta/2), z_k\big) = C(D).$$

Because $\mathrm{diam}(V_{\ell_k}(z_k)) \geqslant d_0$, $V_{\ell_k}(z_k)$ contains a round disk of definite size centered at y_k. Therefore, there is a constant $r_0 = r_0(d_0, D)$ such that $V_{\ell_k}(z_k) \supset B(y_\infty, r_0)$ for large k. Therefore, $f^{\ell_k}(B(y_\infty, r_0)) \subset B(z_k, \delta/2) \subset B(z_\infty, \delta)$. But, this contradicts the fact that $f^{\ell_k}(B(y_\infty, r_0)) \supset J(f)$ when k is large.

Step 2. *There are two constants $L > 0$ and $\nu \in (0, 1)$ such that for any $z \in \partial B$ and any $k \geqslant 1$,* $\mathrm{diam}(V_k(z)) \leqslant L\nu^k$.

By Step 1, there is an integer $s > 0$ such that $\mathrm{diam}(V_s(z)) < \delta/4$ for all $z \in \partial B$. For each $x \in \partial B$, we take a point $x_{kx} \in V_{ks}(x) \cap f^{-ks}(x)$. (Notice that, in general, $V_{ks}(x) \cap f^{-ks}(x)$ consists of finitely many points, x_{ks} can be either of them.) For $0 \leqslant j \leqslant k$, let $x_{js} = f^{(k-j)s}(x_{ks})$ and U_j be the component of $f^{-js}(B(x_{(k-j)s}, \delta/2))$ containing x_{ks}. Then,

$$x_{ks} \in V_{ks}(x) = U_k \subset \cdots \subset U_0 = B(x_{ks}, \delta/2).$$

For every $1 \leqslant j < k$, $f^{js} : U_j \to B(x_{(k-j)s}, \delta/2)$ is a proper map of degree $\leqslant D$. Because $f^{js}(U_{j+1})$ is contained in $B(x_{(k-j)s}, \delta/4)$,

$$\mathrm{mod}(U_j \setminus \overline{U_{j+1}}) \geqslant \frac{1}{D}\mathrm{mod}\big(B(x_{(k-j)s}, \delta/2) \setminus \overline{f^{js}(U_{j+1})}\big) \geqslant \frac{\log 2}{2\pi D},$$
$$\mathrm{mod}\big(B(x_{ks}, \delta/2) \setminus \overline{V_{ks}(x)}\big) \geqslant \sum_{0 \leqslant j < k} \mathrm{mod}(U_j \setminus \overline{U_{j+1}}) \geqslant \frac{k \log 2}{2\pi D}.$$

We know from the proof of Step 1 that $\mathrm{Shape}(V_{ks}(x), x_{ks}) \leqslant C(D)$. Therefore, there is a constant $K(D) > 0$ such that $\min_{y \in \partial V_{ks}(x)} |x_{ks} - y| \geqslant K(D)\mathrm{diam}(V_{ks}(x))$. We have

$$\mathrm{mod}\big(B(x_{ks}, \delta/2) \setminus \overline{V_{ks}(x)}\big) \leqslant \frac{1}{2\pi} \log\left(\frac{\delta}{2K(D)\mathrm{diam}(V_{ks}(x))}\right).$$

It turns out that $\mathrm{diam}(V_{ks}(x)) \leqslant \frac{\delta}{2K(D)} 2^{-k/D}$, which implies that there are two constants $L > 0$ and $\nu \in (0, 1)$ such that $\mathrm{diam}(V_k(x)) \leqslant L\nu^k$ for all $k \geqslant 1$.

Step 3. *There exists a sequence of Jordan curves $\{\gamma_k : \mathbb{S} \to B\}$ such that γ_k converges uniformly to a continuous and surjective map $\gamma_\infty : \mathbb{S} \to \partial B$, where $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ is the unit circle. Hence, ∂B is locally connected.*

Recall that the Böttcher map $\phi : B \to \bar{\mathbb{C}} \setminus \bar{\mathbb{D}}$ defined by $\phi(z) = \lim_{k\to\infty}(f_\lambda^k(z))^{d^{-k}}$ is a conformal isomorphism, which satisfies $\phi^{-1}(r^d e^{2\pi i d t}) = f(\phi^{-1}(re^{2\pi i t}))$ for $(r, t) \in (1, +\infty) \times \mathbb{S}$. Let $\ell(R, t) = \phi^{-1}([\sqrt[d]{R}, R]e^{2\pi i t})$ for $(R, t) \in (1, 2) \times \mathbb{S}$. By the boundary behavior of hyperbolic metric, there is a constant $C > 0$ such that for any$(R, t) \in (1, 2) \times \mathbb{S}$,

$$\begin{aligned}\mathrm{Eucl.length}\big(\ell(R, t)\big) &\leqslant C\mathrm{Hyper.length}\big(\ell(R, t)\big) \cdot \mathrm{H.dist}\big(\phi^{-1}(R\mathbb{S}), \partial B\big)\\ &\leqslant C(\log d)\mathrm{H.dist}\big(\phi^{-1}(R\mathbb{S}), \partial B\big) \quad (\to 0 \text{ as } R \to 1),\end{aligned}$$

where Hyper.length is the hyperbolic length in B and H.dist is the Hausdorff distance in the sphere $\bar{\mathbb{C}}$. Thus, we can choose R sufficiently close to 1 such that for any $t \in \mathbb{S}$, $\ell(R, t) \subset B(z, \delta/2)$ for some $z \in \partial B$. For $k \geqslant 0$, define a curve $\gamma_k : \mathbb{S} \to B$ by $\gamma_k(t) = \phi^{-1}(R^{1/d^k} e^{2\pi i t})$.

Because $f^k(\gamma_{k+q}(t)) = \gamma_q(d^k t)$ for $q \geqslant 0$ and $\gamma_0(d^k t), \gamma_1(d^k t) \in \ell(R, d^k t) \subset B(z, \delta/2)$ for some $z \in \partial B$, we conclude that $\gamma_k(t)$ and $\gamma_{k+1}(t)$ lie in the same component of $f^{-k}(B(z, \delta/2))$ that intersect with ∂B. Based on Step 2,

$$\max_{t\in\mathbb{S}}\big|\gamma_{k+1}(t) - \gamma_k(t)\big| = \mathcal{O}\big(\nu^k\big).$$

So $\{\gamma_k : \mathbb{S} \to B\}$ is a Cauchy sequence and hence converges to a continuous map $\gamma_\infty : \mathbb{S} \to \partial B$.

To finish, we show γ_∞ is surjective. Let $B_k \subset B$ be the disk bounded by $\gamma_k(\mathbb{S})$; then $B_k \Subset B_{k+1}$ and $\bigcup_k B_k = B$. Each point $z \in \partial B$ can therefore be approximated by a sequence of points $\{z_k = \gamma_k(t_k)\}_{k\geqslant 1}$ with $z_k \in \partial B_k$. There is a subsequence k_j such that $t_{k_j} \to t_\infty \in \mathbb{S}$ as $j \to \infty$. We then have $\gamma_\infty(t_\infty) = \lim_j \gamma_{k_j}(t_\infty) = \lim_j \gamma_{k_j}(t_{k_j}) = z$. It follows that γ_∞ is surjective.

Step 4. *If, furthermore, ∂B is a Jordan curve, then ∂B is a quasi-circle.*

Because ∂B is a Jordan curve, the Böttcher map $\phi : B \to \bar{\mathbb{C}} \setminus \bar{\mathbb{D}}$ can be extended to a homeomorphism $\phi : \overline{B} \to \bar{\mathbb{C}} \setminus \mathbb{D}$. Define a map $\psi : \mathbb{S} \to \partial B$ by $\psi(\zeta) = \phi^{-1}(\zeta)$ for $\zeta \in \mathbb{S}$. Then $f(\psi(\zeta)) = \psi(\zeta^d)$. Let $\varphi = \phi|_{\partial B}$ be the inverse of ψ. Both ψ and φ are uniformly continuous; thus, for any sufficiently small positive number ε, there are two small constants $a(\varepsilon), b(\varepsilon)$ such that

$$\forall(\zeta_1, \zeta_2) \in \mathbb{S} \times \mathbb{S}, \qquad |\zeta_1 - \zeta_2| < a(\varepsilon) \quad \Rightarrow \quad \big|\psi(\zeta_1) - \psi(\zeta_2)\big| < \varepsilon,$$
$$\forall(z_1, z_2) \in \partial B \times \partial B, \qquad |z_1 - z_2| < b(\varepsilon) \quad \Rightarrow \quad \big|\varphi(z_1) - \varphi(z_2)\big| < a(\varepsilon).$$

Given two points $z_1, z_2 \in \partial B$, $\partial B \setminus \{z_1, z_2\}$ consists of two components, say E_1 and E_2. Let $L(z_1, z_2) \in \{\overline{E}_1, \overline{E}_2\}$ be a section of ∂B such that $\operatorname{diam}(L(z_1, z_2)) = \min\{\operatorname{diam}(E_1), \operatorname{diam}(E_2)\}$. Thus, for any positive number $\varepsilon \ll \operatorname{diam}(\partial B)$, by uniform continuity we have

$$|z_1 - z_2| < b(\varepsilon) \quad \Rightarrow \quad \operatorname{diam}\big(L(z_1, z_2)\big) < \varepsilon. \tag{1}$$

Based on Alhfors' characterization of quasi-circles [1], to prove that ∂B is a quasi-circle, it suffices to show that there is a constant $C > 0$ such that for any $z_1, z_2 \in \partial B$ with $z_1 \neq z_2$, $\Delta(L(z_1, z_2); z_1, z_2) \leqslant C$. In fact, if $|z_1 - z_2| \geqslant \epsilon$ for some positive constant ϵ, then $\Delta(L(z_1, z_2); z_1, z_2) \leqslant \operatorname{diam}(\partial B)/\epsilon$. Therefore, we only need to consider the case when $|z_1 - z_2|$ is small. In the following, we assume $\delta \ll \operatorname{diam}(\partial B)$ and $|z_1 - z_2| \leqslant b(\delta/2)$; it turns out that $\operatorname{diam}(L(z_1, z_2)) < \delta/2$.

Because f is expanding on ∂B, there is an integer $N > 0$ such that $f^k(L(z_1, z_2)) = \partial B$ for all $k \geqslant N$. We can therefore find a smallest integer $\ell \geqslant 0$ such that

$$\operatorname{diam}\big(f^\ell\big(L(z_1, z_2)\big)\big) < \delta/2, \qquad \operatorname{diam}\big(f^{\ell+1}\big(L(z_1, z_2)\big)\big) \geqslant \delta/2.$$

On the other hand, there exist two points $w_1, w_2 \in f^\ell(L(z_1, z_2))$ such that

$$\begin{aligned}\operatorname{diam}\big(f^{\ell+1}\big(L(z_1, z_2)\big)\big) &= \big|f(w_1) - f(w_2)\big| \leqslant \int_{[w_1, w_2]} \big|f'(z)\big|\,|dz| \\ &\leqslant M|w_1 - w_2| \leqslant M\operatorname{diam}\big(f^\ell\big(L(z_1, z_2)\big)\big),\end{aligned}$$

where $[w_1, w_2]$ is the straight segment connecting w_1 with w_2 and

$$M = \max\{|f'(z)|;\ \text{Eucl.dist}(z, \partial B) \leqslant \delta/2\}.$$

Thus, we have

$$\frac{\delta}{2M} \leqslant \operatorname{diam}\bigl(f^\ell\bigl(L(z_1, z_2)\bigr)\bigr) = \operatorname{diam}\bigl(L\bigl(f^\ell(z_1), f^\ell(z_2)\bigr)\bigr) < \frac{\delta}{2}.$$

By (1), there is a constant $c(\delta, M) > 0$ such that $|f^\ell(z_1) - f^\ell(z_2)| \geqslant c(\delta, M)$.

Applying Lemma 6.1 to the situation $(V_1, U_1) = (U_\ell(f^\ell(z_1)), V_\ell(f^\ell(z_1))), (V_2, U_2) = (B(f^\ell(z_1), \delta), B(f^\ell(z_1), \delta/2))$ and $g = f^\ell$, we conclude that there is a constant $C(D) > 0$ such that

$$\Delta\bigl(L(z_1, z_2);\ z_1, z_2\bigr) \leqslant C(D)\Delta\bigl(f^\ell\bigl(L(z_1, z_2)\bigr);\ f^\ell(z_1), f^\ell(z_2)\bigr) \leqslant \frac{C(D)\delta}{2c(\delta, M)}.$$

Thus, for any $x, y \in \partial B$ with $x \neq y$, the turning $\Delta(L(x, y); x, y)$ is bounded by

$$\max\left\{\frac{\operatorname{diam}(\partial B)}{b(\delta/2)}, \frac{C(D)\delta}{2c(\delta, M)}\right\}. \qquad \square$$

Remark 6.1. Using the same argument as [4], one can show further that if f satisfies **BD** condition on ∂B, then ∂B is a John domain.

The following describes an important case in which f satisfies the **BD** condition on ∂B.

Proposition 6.2. *If $\#(P(f) \cap \partial B) < \infty$ and all periodic points in $P(f) \cap \partial B$ are repelling, then f satisfies* **BD** *condition on ∂B.*

Proof. The proof is based on the following claim.

Claim. *For any $u \in \partial B$, there is a constant $\varepsilon_u > 0$ such that for any $m \geqslant 0$ and any component $U_m(u)$ of $f^{-m}(B(u, \varepsilon_u))$ that intersects with ∂B, $U_m(u)$ contains at most one critical point of f^m.*

The claim implies that $U_m(u)$ is simply connected by the Riemann–Hurwitz formula. Because the sequence $U_m(u) \to f(U_m(u)) \to \cdots \to f^{m-1}(U_m(u)) \to B(u, \varepsilon_u)$ meets every critical point of f at most once, we conclude that $\deg(f^m : U_m(u) \to B(u, \varepsilon_u))$ is bounded by $D = \prod_{c \in C(f)} \deg(f, c)$.

In the following, we prove the claim.

First, note that every point in $P(f) \cap \partial B$ is pre-periodic; we can deconstruct ∂B into three disjoint sets X, Y and Z, where $X = \partial B \setminus P(f)$, Z is the union of all repelling cycles in $P(f) \cap \partial B$ and $Y = (P(f) \cap \partial B) \setminus Z$.

For any $x \in X$, choose a small number $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \cap P(f) = \emptyset$. Then, for any component $W_m(x)$ of $f^{-m}(B(x, \varepsilon_x))$ intersecting with ∂B, $f^m : W_m(x) \to B(x, \varepsilon_x)$ is a conformal map.

The set Y consists of all strictly pre-periodic points. Thus, there is an integer $q \geqslant 1$ such that for any $y \in Y$, $f^{-q}(y) \cap P(f) \cap \partial B = \emptyset$. For an open set U in $\bar{\mathbb{C}}$ and a point $u \in U$, we use $\mathrm{Comp}_u(U)$ to denote the component of U that contains u. For every $y \in Y$, choose $\varepsilon_y > 0$

small enough such that for any $x \in f^{-q}(y) \cap \partial B \subset X$, $\mathrm{Comp}_x(f^{-q}(B(y, \varepsilon_y))) \subset B(x, \varepsilon_x)$ and $\mathrm{Comp}_x(f^{-q}(B(y, \varepsilon_y)))$ contain at most one critical point of f^q.

Finally, we deal with Z. For $z \in Z$, suppose z lies in a repelling cycle of period p. Choose $\varepsilon_z > 0$ such that

(1) $B(z, \varepsilon_z)$ is contained in the linearizable neighborhood of z and $\mathrm{Comp}_z(f^{-p}(B(z, \varepsilon_z)))$ is a subset of $B(z, \varepsilon_z)$.

(2) For every $u \in (f^{-p}(z) \cap \partial B) \setminus \{z\} \subset X \cup Y$, $\mathrm{Comp}_u(f^{-p}(B(z, \varepsilon_z)))$ contains at most one critical point of f^p and $\mathrm{Comp}_u(f^{-p}(B(z, \varepsilon_z))) \subset B(u, \varepsilon_u)$.

One can easily verify that the collection of neighborhoods $\{B(u, \varepsilon_u),\ u \in \partial B\}$ are just as required. □

Corollary 6.1. *If f is critically finite, then f satisfies the* **BD** *condition on ∂B.*

Proof. Because f is critically finite, every periodic point of f is either repelling or superattracting, which implies that $\#(P(f) \cap \partial B) < \infty$ and all periodic points in $P(f) \cap \partial B$ are repelling. Thus, by Proposition 6.2, f satisfies the **BD** condition on ∂B. □

7. The boundary ∂B_λ is a Jordan curve

In this section, we will prove Theorem 1.1 and Theorem 1.2. The strategy of the proof is as follows.

First, consider the McMullen maps f_λ with parameter $\lambda \in \mathcal{H}$. If f_λ is critically finite, then the Julia set is locally connected. Otherwise, by Proposition 4.1, we can find an admissible graph $\mathbf{G}_\lambda(\theta_1, \ldots, \theta_N)$. With respect to the Yoccoz puzzle induced by this graph, there are two possibilities:

Case 1. None of $T(c)$ with $c \in C_\lambda$ is periodic. This case is discussed in Section 7.1, and the local connectivity of $J(f_\lambda)$ follows from Proposition 7.1. The idea of the proof is based on the combinatorial analysis for tableaux introduced by Branner and Hubbard (see [2,21]) and on 'modified puzzle piece' techniques.

Case 2. Some $T(c)$ with $c \in C_\lambda$ is periodic. In this case, the map f_λ is either renormalizable or $*$-renormalizable. This case is discussed in Section 7.2. The local connectivity of ∂B_λ follows from Proposition 7.2. The goal of the proof of Proposition 7.2 is to construct a closed curve separating ∂B_λ from the small filled Julia set K_c.

In Section 7.3, we deal with the real parameters $\lambda \in \mathbb{R}^+$.

In Section 7.4, we improve the regularity of the boundary ∂B_λ. We first include a proof of Devaney that claims that the local connectivity of ∂B_λ implies that ∂B_λ is a Jordan curve. We then show that ∂B_λ is a quasi-circle except in two specific cases.

In Section 7.5, we present some corollaries.

7.1. None of $T(c)$ with $c \in C_\lambda$ is periodic

Recall that J_0 is the set of all points on the Julia set $J(f_\lambda)$ whose orbits eventually meet the graph $\mathbf{G}_\lambda(\theta_1, \ldots, \theta_N)$.

Lemma 7.1. *Let $z \in J(f_\lambda) \setminus J_0$. If $T(z)$ is non-critical, then* $\mathrm{End}(z) := \bigcap_{d \geqslant 0} \overline{P_d(z)} = \{z\}$.

Proof. It suffices to prove $\mathrm{End}(f_\lambda(z)) = \{f_\lambda(z)\}$. Because $T(z)$ is non-critical, there is an integer $d_0 \geqslant 1$ such that for any $j > 0$, the position (d_0, j) is not critical. Equivalently, for any $d \geqslant d_0$ and any $j \geqslant 1$, the puzzle piece $P_d(f_\lambda^j(z))$ contains no critical point. Let $\{\hat{P}_{d_0-1}^{(i)}$ and $1 \leqslant i \leqslant M\}$ be the collection of all modified puzzle pieces of depth $d_0 - 1$, numbered so that $\hat{P}_{d_0-1}^{(1)} = \hat{P}_{d_0-1}(v_\lambda^+)$, $\hat{P}_{d_0-1}^{(2)} = \hat{P}_{d_0-1}(v_\lambda^-)$, and recall that we use $\hat{P}_d(w)$ to denote the modified puzzle piece of $P_d(w)$. Every modified puzzle piece of depth $\geqslant d_0$ is contained in a unique modified puzzle piece $\hat{P}_{d_0-1}^{(i)}$ of depth $d_0 - 1$. Let $\mathrm{dist}_i(x, y)$ be the Poincaré metric of $\hat{P}_{d_0-1}^{(i)}$. For $2 < i \leqslant M$, there are exactly $2n$ branches of f_λ^{-1} on $\hat{P}_{d_0-1}^{(i)}$, say $g_1^i, g_2^i, \ldots, g_{2n}^i$, and each g_k^i on $\hat{P}_{d_0-1}^{(i)}$ is univalent and carries $\hat{P}_{d_0}^{(\alpha)} \Subset \hat{P}_{d_0-1}^{(i)}$ onto a proper subset of some $\hat{P}_{d_0-1}^{(j)}$. It follows that there is a uniform constant $0 < \nu < 1$ such that

$$\mathrm{dist}_j\big(g_k^i(x), g_k^i(y)\big) \leqslant \nu \mathrm{dist}_i(x, y)$$

for any $x, y \in \hat{P}_{d_0}^{(\alpha)} \Subset \hat{P}_{d_0-1}^{(i)}$ and any $2 < i \leqslant M$, $1 \leqslant k \leqslant 2n$.

Let D be the maximum Poincaré diameters of the modified puzzle pieces of depth d_0. For any integer $h > 0$, because the sequence

$$P_{d_0+h}\big(f_\lambda(z)\big) \to P_{d_0+h-1}\big(f_\lambda^2(z)\big) \to \cdots \to P_{d_0+1}\big(f_\lambda^h(z)\big) \to P_{d_0}\big(f_\lambda^{h+1}(z)\big)$$

contains no critical point (this follows from the assumption that $T(z)$ is non-critical), it follows that

$$\mathrm{Hyper.diam}\big(P_{d_0+h}\big(f_\lambda(z)\big)\big) \leqslant D\nu^h$$

with respect to the Poincaré metric of $\hat{P}_{d_0-1}(f_\lambda(z))$. Thus, we have $\mathrm{End}(f_\lambda(z)) = \{f_\lambda(z)\}$. $\square$

Proposition 7.1. *If $T(c)$ is not periodic for any $c \in C_\lambda$, then the Julia set $J(f_\lambda)$ is locally connected.*

Proof. Note that $T(c)$ is either critical or non-critical. First, we prove $\mathrm{End}(c) = \{c\}$ and $\mathrm{End}(z) = \{z\}$ for any $z \in J(f_\lambda) \setminus J_0$. We then deal with the points that lie in J_0.

Case 1. $T(c)$ is critical. Because the graph is admissible, we can find a non-degenerate annulus $A_{d_0}(c)$. Consider the descendents of $\mathrm{Row}_c(d_0)$. It is obvious that if $\mathrm{Row}_c(t)$ is a descendent in the k-th generation of $\mathrm{Row}_c(d_0)$, the annulus $A_t(c)$ is non-degenerate with modulus $\mathrm{mod}(A_{d_0}(c))/2^k$. If $\mathrm{Row}_c(d_0)$ has at least 2^k descendents in the k-th generation for each $k \geqslant 1$, then each of these contributes exactly $\mathrm{mod}(A_{d_0}(c))/2^k$ to the sum $\sum_d \mathrm{mod}(A_d(c))$. Hence, $\sum_d \mathrm{mod}(A_d(c)) = \infty$, as required. On the other hand, if there are fewer descendents in some generation, then one of them, say $\mathrm{Row}_c(m)$, must be an only child, hence excellent by Lemma 4.7. Again by Lemma 4.7, we see that $\sum_d \mathrm{mod}(A_d(c)) = \infty$. Therefore, in either case, $\mathrm{End}(c) = \{c\}$.

Now consider a point $z \in J(f_\lambda) \setminus (J_0 \cup C_\lambda)$. If $T(z)$ is non-critical, then by Lemma 7.1, $\mathrm{End}(z) = \{z\}$. If $T(z)$ is critical, then for each $d \geqslant 1$, there is a smallest integer $l_d \geqslant 0$ such that both (d, l_d) and (d, l_d+1) are critical positions. It follows that $f_\lambda^{l_d} : A_{d+l_d}(z) \to A_d(c')$ is a conformal map for some $c' \in C_\lambda$. In this case, $\sum_d \mathrm{mod}(A_d(z)) \geqslant \sum_d \mathrm{mod}(A_{d+l_d}(z)) = \sum_d \mathrm{mod}(A_d(c)) = \infty$, hence $\mathrm{End}(z) = \{z\}$.

Case 2. $T(c)$ is non-critical. It follows from Lemma 7.1 that $\mathrm{End}(c) = \{c\}$. For $z \in J(f_\lambda) \setminus (J_0 \cup C_\lambda)$, we assume $T(z)$ is critical; otherwise, $\mathrm{End}(z) = \{z\}$ based on Lemma 7.1. Suppose $A_{d_0}(c)$ is a non-degenerate annulus and $(d_0+1, l_1), (d_0+1, l_2), \ldots$ are all critical positions in the (d_0+1)-th row of the tableau $T(z)$. Because all tableaux $T(c)$ with $c \in C_\lambda$ are non-critical, there is a constant D such that $\deg(f_\lambda^{l_k} : P_{d_0+l_k}(z) \to P_{d_0,l_k}(z)) \leqslant D$ for all $k \geqslant 1$. Thus,

$$\mathrm{mod}\big(A_{d_0+l_k}(z)\big) \geqslant D^{-1}\mathrm{mod}\big(A_{d_0}(c)\big)$$

for all $k \geqslant 1$. Hence, $\sum_d \mathrm{mod}(A_d(z)) \geqslant \sum_k \mathrm{mod}(A_{d_0+l_k}(z)) = \infty$ and $\mathrm{End}(z) = \{z\}$.

Points that lie in J_0**.** For any $z \in J_0$, the orbit $z \mapsto f_\lambda(z) \mapsto f_\lambda^2(z) \mapsto \cdots$ eventually meets the graph $\mathbf{G}_\lambda(\theta_1, \ldots, \theta_N)$. Therefore, the Euclidean distance between the critical set C_λ and the orbit $\{f_\lambda^k(z)\}_{k\geqslant 0}$ is bounded below by some positive number $\epsilon(z)$. In addition, for every d large enough, z lies in the common boundary of exactly two puzzle pieces of depth d. We denote these two puzzle pieces by $P'_d(z)$ and $P''_d(z)$. In the previous argument, we have already proved that $\mathrm{End}(c) = \{c\}$; this implies $\mathrm{Eucl.diam}(P_d(c)) \to 0$ as $d \to \infty$. Choose a d_0 large enough such that

$$\mathrm{Eucl.diam}\big(P_{d_0}(c)\big) < \epsilon(z) \leqslant \mathrm{Eucl.dist}\big(C_\lambda, \big\{f_\lambda^k(z)\big\}_{k\geqslant 0}\big).$$

Then, the orbit $z \mapsto f_\lambda(z) \mapsto f_\lambda^2(z) \mapsto \cdots$ avoids all the critical puzzle pieces of depth d_0. Let $P_d^*(z) = \overline{P'_d(z) \cup P''_d(z)}$ for d large enough. Then, the proof of Lemma 7.1 applies equally well to this situation, and $\bigcap_d P_d^*(z) = \{z\}$ immediately follows.

Connectivity of neighborhoods. Let

$$P_d^*(z) = \begin{cases} \overline{P_d(z)}, & \text{if } z \in J(f_\lambda) \setminus J_0, \\ \overline{P'_d(z) \cup P''_d(z)}, & \text{if } z \in J_0 \text{ and } d \text{ is large.} \end{cases}$$

Based on Lemma 4.2, for every $z \in J(f_\lambda)$ and every large integer d, the intersection $P_d^*(z) \cap J(f_\lambda)$ is a connected and compact subset of $J(f_\lambda)$. Thus, $\{P_d^*(z) \cap J(f_\lambda)\}$ forms a basis of connected neighborhoods of z. Because $\bigcap(P_d^*(z) \cap J(f_\lambda)) = \{z\}$, the Julia set is locally connected at z. Note that z is arbitrarily chosen, we conclude that $J(f_\lambda)$ is locally connected. □

7.2. *Some $T(c)$ with $c \in C_\lambda$ is periodic*

Suppose some tableau $T(c)$ with $c \in C_\lambda$ is k-periodic for some $k > 0$. Based on the proof of Lemma 5.1, f_λ is either k-renormalizable at c or $k/2$-$*$-renormalizable at c. Let $(\epsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$, where d_0 is a large integer, be the renormalization and

$$(\epsilon, p) = \begin{cases} (1, k), & \text{if } f_\lambda \text{ is } k\text{-renormalizable at } c, \\ (-1, k/2), & \text{if } f_\lambda \text{ is } k/2\text{-}*\text{-renormalizable at } c. \end{cases}$$

The small filled Julia set of the renormalization $(\epsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is denoted by K_c. Recall that β_c is the β-fixed point of the renormalization and β'_c is the other preimage of β_c under the map $\epsilon f_\lambda^p|_{P_{d_0+p}(c)}$.

Assume now that $K_c \cap \partial B_\lambda \neq \emptyset$; then, based on Lemma 5.3, $\beta_c \in K_c \cap \partial B_\lambda$ and there is a unique external ray, say $R_\lambda(\theta)$, landing at β_c. The angle θ is of the form $\frac{m}{2^k-1}$. It follows that $\beta'_c \in K_c \cap \partial T_\lambda$ and there is a unique radial ray $R_{T_\lambda}(\alpha_\theta)$ in T_λ landing at β'_c. The radial ray $R_{T_\lambda}(\alpha_\theta)$ satisfies $\epsilon f_\lambda^p(R_{T_\lambda}(\alpha_\theta)) = R_\lambda(\theta)$. Let

$$K = K_c \cup \overline{R_\lambda(\theta)} \cup \overline{R_{T_\lambda}(\alpha_\theta)} \cup (-K_c) \cup \left(-\overline{R_\lambda(\theta)}\right) \cup \left(-\overline{R_{T_\lambda}(\alpha_\theta)}\right).$$

The set K is a connected and compact subset of $\bar{\mathbb{C}}$. Note that $-R_{T_\lambda}(\alpha_\theta) = R_{T_\lambda}(\alpha_\theta + 1/2)$. Let Δ_1 be the component of $\bar{\mathbb{C}} \setminus (K \cup \overline{B_\lambda})$ that intersects with $Q_{T_\lambda}(\alpha_\theta, \alpha_\theta + 1/2)$ and Δ_2 be the component of $\bar{\mathbb{C}} \setminus (K \cup \overline{B_\lambda})$ that intersects with $Q_{T_\lambda}(\alpha_\theta + 1/2, \alpha_\theta)$, where we use $Q_{T_\lambda}(\theta_1, \theta_2)$ to denote the set $\{\phi_{T_\lambda}(re^{2\pi it});\ 0 < r < 1,\ \theta_1 \leqslant t \leqslant \theta_2\}$. Because $K \cup \overline{B_\lambda}$ is connected and compact, both Δ_1 and Δ_2 are disks. Let Z_i be the component of $\bar{\mathbb{C}} \setminus K$ that contains Δ_i.

The aim of this section is to prove:

Proposition 7.2. *Assume that $K_c \cap \partial B_\lambda \neq \emptyset$, then for $i \in \{1,2\}$, there is a curve $\mathcal{L}_i \subset \Delta_i \cup \{0\}$ stemming from T_λ and converging to β_c. More precisely, $\mathcal{L}_i$ can be parameterized as $\mathcal{L}_i : [0,+\infty) \to \Delta_i \cup \{0\}$ such that $\mathcal{L}_i(0) = 0$, $\mathcal{L}_i((0,+\infty)) \subset \Delta_i$ and $\lim_{t\to+\infty} \mathcal{L}_i(t) = \beta_c$ (Fig. 13).*

Proof. Let $\Gamma = \bigcup_{j\geqslant 0}(\pm f_\lambda^j(K_c \cup \overline{R_\lambda(\theta)}))$. By Lemma 5.3, any two distinct elements in the set $\{\pm f_\lambda^j(K_c \cup \overline{R_\lambda(\theta)});\ j \geqslant 0\}$ intersect only at the point ∞, which implies that $U = \bar{\mathbb{C}} \setminus \Gamma$ is a disk.

Step 1. *There exists $G_i : U \to U \cap Z_i$, an inverse branch of ϵf_λ^p such that the sequence $\{G_i^l;\ l \geqslant 0\}$ converges locally and uniformly in U to a constant $z_i \in K_c$.*

Because U has no intersection with the post-critical set of f_λ, its preimage $f_\lambda^{-1}(U)$ has exactly $2n$ components, say $V_1, \ldots, V_{2n}$. These components are arranged symmetrically about the origin under the rotation $z \mapsto e^{\pi i/n}z$. For every $1 \leqslant j \leqslant 2n$, $f_\lambda : V_j \to U$ is a conformal map. Moreover, $f_\lambda^{-1}(U) \subset \bar{\mathbb{C}} \setminus K$.

For $1 \leqslant j \leqslant p-1$, let $\Omega_j \in \{V_1, \ldots, V_{2n}\}$ be the component of $f_\lambda^{-1}(U)$ such that $\overline{\Omega}_j \cap f_\lambda^j(K_c) \neq \emptyset$ and the inverse of $f_\lambda : \Omega_j \to U$ is denoted by g_j. For $j = 0$, let Ω_0^i be the component of $f_\lambda^{-1}(U)$ such that $\overline{\Omega_0^i} \cap K_c \neq \emptyset$ and $\Omega_0^i \subset Z_i$. The inverse of $f_\lambda : \Omega_0^i \to U$ is denoted by g_0^i for $i \in \{1,2\}$.

Now, we define

$$G_i(z) = \begin{cases} g_0^i \circ g_1 \circ \cdots \circ g_{p-1}(\epsilon z),\ z \in U & \text{if } p \geqslant 2, \\ g_0^i(\epsilon z),\ z \in U & \text{if } p = 1. \end{cases}$$

Because $(\epsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is a p-($*$-)renormalization of f_λ at c, we have $G_i(P_{d_0}(c) \cap U) \subset P_{d_0+p}(c) \cap Z_i$. The map $G_i : U \to U$ is not surjective; thus, by the Denjoy–Wolff theorem (see [20]), the sequence $\{G_i^l;\ l \geqslant 0\}$ converges locally and uniformly in U to a constant z_i. It follows from $G_i(P_{d_0}(c) \cap U) \subset P_{d_0+p}(c) \cap Z_i$ that $z_i \in K_c$.

Step 2. *There exists a curve $C_i \subset U \cap (\Delta_i \cup \{0\})$ connecting 0 with $G_i(0)$ for $i \in \{1,2\}$.*

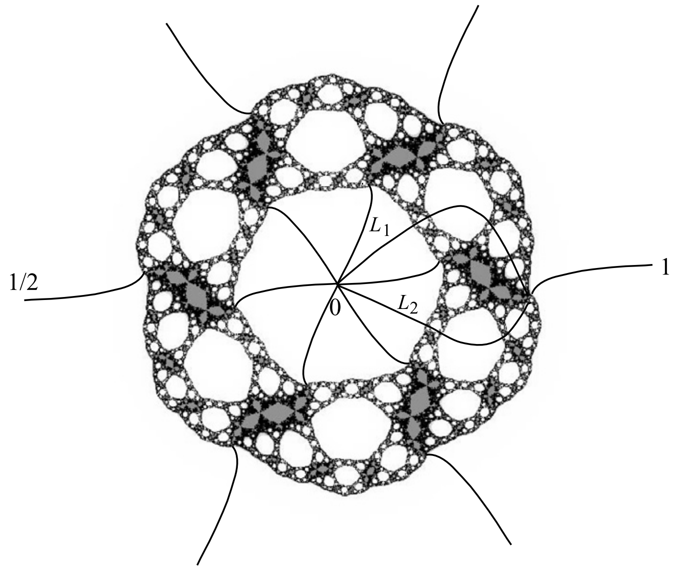

Fig. 13. Constructing two curves L_1 and L_2 that converge to β_c, here $n = 3$ and f_λ is 1-renormalizable at $c = c_0$.

Because the graph $\mathbf{G}_\lambda(\theta_1, \ldots, \theta_N)$ is admissible, the filled Julia set K_c is disjointed from the boundary of any puzzle piece. Thus, for any $\alpha \in \{\tau^s(\theta_j);\ 1 \leqslant j \leqslant N,\ s \geqslant 0\}$, Γ is disjoint from the cut ray Ω_λ^α outside ∞. (This is because the external ray $R_\lambda(\theta)$ has no intersection with $g_\lambda(\theta_1, \ldots, \theta_N)$ outside ∞; compare Lemma 5.3.) By Proposition 4.2, for any angle $\alpha \in \{\tau^s(\theta_j);\ 1 \leqslant j \leqslant N,\ s \geqslant 0\}$ and any map $g \in \{g_0^1, g_0^2, g_1, \ldots, g_{p-1}\}$, only one curve of $g(\omega_\lambda^\alpha \setminus \{\infty\}), g(\omega_\lambda^{\alpha+1/2} \setminus \{\infty\})$ intersects with ∂B_λ, and the other curve connects 0 with a preimage of 0.

Fix an angle $\alpha \in \{\tau^s(\theta_j);\ 1 \leqslant j \leqslant N,\ s \geqslant 0\}$; we define a curve family $\mathcal{F}$ by

$$\mathcal{F} = \left\{\epsilon\omega_\lambda^\alpha \setminus \{\infty\};\ \epsilon^{2n} = 1 \text{ and } \epsilon\omega_\lambda^\alpha \subset \bigcup_{j \in \mathbb{I} \setminus \{0,n\}} S_j\right\}.$$

We construct the curve C_i by an inductive procedure, as follows:

First, choose a curve $\zeta_{p-1} \in \mathcal{F}$ such that $g_{p-1}(\zeta_{p-1}) \cap \partial B_\lambda = \emptyset$ and let $\gamma_{p-1} = g_{p-1}(\zeta_{p-1})$. Suppose that for some $2 \leqslant j \leqslant p-1$ we have already constructed the curves $\gamma_{p-1}, \ldots, \gamma_j$. We then choose $\zeta_{j-1} \in \mathcal{F}$ such that $g_{j-1}(\zeta_{j-1}) \cap \partial B_\lambda = \emptyset$ and $\zeta_{j-1} \cap \gamma_j = \emptyset$ and let $\gamma_{j-1} = g_{j-1}(\zeta_{j-1} \cup \gamma_j)$. In this way, we can construct a sequence of curves $\gamma_{p-1}, \gamma_{p-2}, \ldots, \gamma_2, \gamma_1$ step by step, and each curve has no intersection with ∂B_λ. These curves connect 0 with some iterated preimage of 0. By construction,

$$\gamma_1 = \bigcup_{1 \leqslant j \leqslant p-1} g_1 \circ \cdots \circ g_j(\zeta_j).$$

We now choose $\zeta_0^i \in \mathcal{F}$ such that $g_0^i(\zeta_0^i) \cap \partial B_\lambda = \emptyset$ and $\zeta_0^i \cap \gamma_1 = \emptyset$, and let

$$C_i = \begin{cases} g_0^i(\zeta_0^i \cup \gamma_1) \cup \{0\}, & \text{if } p \geqslant 2, \\ g_0^i(\zeta_0^i) \cup \{0\}, & \text{if } p = 1. \end{cases}$$

The curve C_i connects 0 to $G_i(0)$ and $C_i \subset U \cap (\Delta_i \cup \{0\})$, as required.

Step 3. *The union $\mathcal{L}_i = \bigcup_{j \geqslant 0} G_i^j(C_i)$ is the curve contained in $\Delta_i \cup \{0\}$ and converging to β_c.* By construction, $G_i(\mathcal{L}_i) \subset G_i(\mathcal{L}_i) \cup C_i = \mathcal{L}_i$ and $\mathcal{L}_i \setminus \{0\} \subset \Delta_i$.

To finish, we show $\mathcal{L}_i$ converges to β_c. By step 1, the sequence $\{G_i^k;\ k \geqslant 0\}$ converges uniformly on any compact subset of U to a constant $z_i \in K_c$. Because C_i is a compact subset of U, the curve $\mathcal{L}_i$ converges to $z_i \in K_c$ and $G_i(z_i) = z_i$. Because $\epsilon f_\lambda^p(\mathcal{L}_i) \supset \mathcal{L}_i$, we conclude $z_i = \beta_c$ by Lemma 5.2. □

Corollary 7.1. *If $T(c)$ is periodic for some $c \in C_\lambda$, then ∂B_λ is locally connected.*

Proof. We can assume that f_λ is not geometrically finite; otherwise, the Julia set is locally connected (see [29]). Thus, f_λ has no parabolic point.

If $K_c \cap \partial B_\lambda = \emptyset$, then for all $j \geqslant 0$, $f_\lambda^j(K_c) \cap \partial B_\lambda = \emptyset$. Because $P(f_\lambda)$ is a subset of $(\bigcup_{j \geqslant 0} f_\lambda^j(\pm f_\lambda(K_c))) \cup \{\infty\}$, we conclude $P(f_\lambda) \cap \partial B_\lambda = \emptyset$. Based on Proposition 6.1 and Proposition 6.2, ∂B_λ is locally connected.

If $K_c \cap \partial B_\lambda \neq \emptyset$, then by Proposition 7.2, the closed curve $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{\beta_c\}$ separates $K_c \setminus \{\beta_c\}$ from $\partial B_\lambda \setminus \{\beta_c\}$. In this case, for all $j \geqslant 0$, $f_\lambda^j(K_c) \cap \partial B_\lambda = \{f_\lambda^j(\beta_c)\}$. Thus, $\#(P(f_\lambda) \cap \partial B_\lambda) < \infty$, and all periodic points in $P(f_\lambda) \cap \partial B_\lambda$ are repelling. Again by Proposition 6.1 and Proposition 6.2, ∂B_λ is locally connected. □

7.3. Real case

In this section, we will deal with real parameters. Due to the symmetry of the parameter plane, we only need to consider $\lambda \in \mathbb{R}^+ = (0, +\infty)$. In this case, the Julia set $J(f_\lambda)$ is symmetric about the real axis. If $C_\lambda \subset A_\lambda$, by 'The Escape Trichotomy' (Theorem 2.1), the Julia set $J(f_\lambda)$ is either a Cantor set, a Cantor set of circles or a Sierpinski curve. In the latter two cases, the local connectivity of ∂B_λ is already known. In the following discussion, we assume $C_\lambda \cap A_\lambda = \emptyset$.

Lemma 7.2. *Suppose $\lambda \in \mathbb{R}^+$ and $C_\lambda \cap A_\lambda = \emptyset$; then, f_λ is 1-renormalizable at $c_0 = \sqrt[2n]{\lambda}$.*

Proof. Let U be the interior of $(S_0 \cup S_{-(n-1)}) \setminus \{z \in B_\lambda \cup T_\lambda;\ G_\lambda(z) \geqslant 1\}$ and $V = \bar{\mathbb{C}} \setminus (\{z \in B_\lambda;\ G_\lambda(z) \geqslant n\} \cup [-\infty, v_\lambda^-])$. One can easily verify that $f_\lambda : U \to V$ is a quadratic-like map. Because $C_\lambda \cap A_\lambda = \emptyset$, the critical orbit $\{f_\lambda^k(c_0);\ k \geqslant 0\}$ is contained in $U \cap \mathbb{R}^+$. This implies that (f_λ, U, V) is a 1-renormalization of f_λ at c_0. □

Let $K_{c_0} = \bigcap_{k \geqslant 0} f_\lambda^{-k}(U)$ be the small filled Julia set of the renormalization (f_λ, U, V), β_{c_0} be the β-fixed point and β'_{c_0} be the preimage of β_{c_0}. It is easy to check that K_{c_0} is symmetric about the real axis and $K_{c_0} \cap \mathbb{R}^+$ is a connected and closed interval.

Proposition 7.3. $K_{c_0} \cap \partial B_\lambda = \{\beta_{c_0}\}$.

Proof. As with the proof of Proposition 7.2, the idea of the proof is to construct a Jordan curve $\mathcal{C}$ that separates $K_{c_0} \setminus \{\beta_{c_0}\}$ from $\partial B_\lambda \setminus \{\beta_{c_0}\}$.

We first show that β_{c_0} is the landing point of the zero external ray $R_\lambda(0)$. Note that rational external rays (i.e., external rays with a rational angle) always land. Let z_0 be the landing point of $R_\lambda(0)$. Obviously, $R_\lambda(0) \subset \mathbb{R}^+$ and z_0 is a fixed point of f_λ, which implies that $z_0 \in U \cap \mathbb{R}^+$, and the orbit of z_0 does not escape from U. Therefore, $z_0 \in K_{c_0}$. Because $R_\lambda(0)$ is an f_λ-invariant ray that lands at z_0, we conclude $z_0 = \beta_{c_0}$ based on Lemma 5.2.

Let $K = K_{c_0} \cup [\beta_{c_0}, +\infty] \cup (-K_{c_0}) \cup [-\infty, -\beta_{c_0}]$. One can easily verify $f_\lambda^{-1}(K) = \bigcup_{\omega^{2n}=1} \omega(K_{c_0} \cup [0, +\infty])$. The set $Y = \bar{\mathbb{C}} \setminus K$ is a disk, and its preimage $f_\lambda^{-1}(Y)$ consists of $2n$ components that are symmetric about the origin under the rotation $z \mapsto e^{i\pi/n} z$. For each component X of $f_\lambda^{-1}(Y)$, $f_\lambda : X \to Y$ is a conformal map. Let X_0 be the component of $f_\lambda^{-1}(Y)$ that is contained in S_0 and g be the inverse map of $f_\lambda : X_0 \to Y$. Based on the Denjoy–Wolff theorem, the sequence of maps $\{g^k;\ k \geqslant 0\}$ converges locally and uniformly in Y to a constant, say x. Because $g(Y \cap V) \subset X_0 \cap U$, we conclude $x \in K_{c_0}$.

Let Δ be the component of $\bar{\mathbb{C}} \setminus (\overline{B}_\lambda \cup K_{c_0} \cup (-K_{c_0}) \cup \mathbb{R})$ that intersects with T_λ and lies in the upper half plane.

Claim. *There is a path $\mathcal{L} \subset \Delta \cup \{0\}$ stemming from T_λ and converging to β_{c_0}. More precisely, $\mathcal{L}$ can be parameterized as $\mathcal{L} : [0, +\infty) \to \Delta \cup \{0\}$ such that $\mathcal{L}(0) = 0$, $\mathcal{L}((0, +\infty)) \subset \Delta$ and* $\lim_{t \to +\infty} \mathcal{L}(t) = \beta_{c_0}$.

Let $p_0 = \sqrt[2n]{-\lambda}$ be the preimage of 0 that lies in S_0 and $\gamma_0 = [0, p_0]$ be the segment connecting 0 with p_0. Then, $\gamma_0 \cap (K_{c_0} \cup \partial B_\lambda) = \emptyset$. Indeed, $\gamma_0 \cap K_{c_0} = \emptyset$ follows from the fact that $f_\lambda(\gamma_0) \cap K_{c_0} \subset i\mathbb{R} \cap K_{c_0} = \emptyset$. In the following, we show that $\gamma_0 \cap \partial B_\lambda = \emptyset$. It suffices to show that $B_\lambda \cap D = \emptyset$, where $D = \{z \in \mathbb{C};\ |z| < \sqrt[2n]{\lambda}\}$. Otherwise, $B_\lambda \cap D \neq \emptyset$ implies $B_\lambda \cap \partial D \neq \emptyset$. Because $\varphi : z \mapsto \sqrt[n]{\lambda}/\bar{z}$ maps B_λ onto T_λ and the restriction $\varphi|_{\partial D}$ is the identity map, we have $B_\lambda \cap \partial D = \varphi(B_\lambda \cap \partial D) = T_\lambda \cap \partial D$. But this implies $B_\lambda \cap T_\lambda \neq \emptyset$, contradiction.

Note that g maps γ_0 outside D and $g(\gamma_0)$ connects p_0 with a preimage of p_0 that lies inside S_0. Let $\mathcal{L} = \bigcup_{k \geqslant 0} g^k(\gamma_0)$. By construction, $\mathcal{L} \cap (K_{c_0} \cup \partial B_\lambda) = \emptyset$, and $\mathcal{L}$ converges to $x \in K_{c_0}$. Because $f_\lambda(\mathcal{L}) = \mathcal{L} \cup f_\lambda(\gamma_0) \supset \mathcal{L}$, we conclude $x = \beta_{c_0}$ based on Lemma 5.2.

Let $\mathcal{C} = \mathcal{L} \cup \mathcal{L}^* \cup \{\beta_{c_0}\}$, where $\mathcal{L}^* = \{\bar{z};\ z \in \mathcal{L}\}$. $\mathcal{C}$ is a Jordan curve separating $K_{c_0} \setminus \{\beta_{c_0}\}$ from $\partial B_\lambda \setminus \{\beta_{c_0}\}$. The conclusion follows. $\square$

Remark 7.1. Based on the proof of Proposition 7.3, we conclude

$$\partial B_\lambda \cap \mathbb{R} = \{\pm\beta_{c_0}\}, \qquad K_{c_0} \cap \mathbb{R} = \left[\beta'_{c_0}, \beta_{c_0}\right], \qquad \partial T_\lambda \cap \mathbb{R} = \left\{\pm\beta'_{c_0}\right\}.$$

Corollary 7.2. *Suppose $\lambda \in \mathbb{R}^+$ and $C_\lambda \cap A_\lambda = \emptyset$; then, ∂B_λ is locally connected.*

Proof. By Proposition 7.3, if n is odd, then $P(f_\lambda) \cap \partial B_\lambda \subset (-K_{c_0} \cup K_{c_0}) \cap \partial B_\lambda \subset \{\pm\beta_{c_0}\}$; if n is even, then $P(f_\lambda) \cap \partial B_\lambda \subset K_{c_0} \cap \partial B_\lambda \subset \{\beta_{c_0}\}$. If β_{c_0} is a parabolic point, then f_λ is geometrically finite, and the local connectivity of ∂B_λ follows from [29]. Otherwise, based on Propositions 6.1 and 6.2, ∂B_λ is also locally connected. $\square$

7.4. Local connectivity implies higher regularity

At this point, we have already proven that ∂B_λ is locally connected if the Julia set is not a Cantor set. Based on the arguments of Devaney [5], we prove the following proposition, which will lead to Theorem 1.1.

Proposition 7.4. *If ∂B_λ is locally connected, then ∂B_λ is a Jordan curve.*

Proof. Let W_0 be the component of $\bar{\mathbb{C}} - \overline{B}_\lambda$ containing 0. It is obvious that $\partial W_0 \subset \partial B_\lambda$, $T_\lambda \subset W_0$, $\partial T_\lambda \subset \overline{W}_0$. Based on Lemma 2.1, $e^{i\pi/n} W_0 = W_0$.

Recall that $H_\lambda(z) = \sqrt[n]{\lambda}/z$, so $H_\lambda(\partial W_0) \subset H_\lambda(\partial B_\lambda) = \partial T_\lambda \subset \overline{W}_0$. Because ∂B_λ is locally connected, ∂W_0 is locally connected. It follows that $\bar{\mathbb{C}} - \overline{W}_0$ is connected and $H_\lambda(\bar{\mathbb{C}} - \overline{W}_0) \subset W_0$.

Now, we show that $f_\lambda^{-1}(0) \subset W_0$. If not, $f_\lambda^{-1}(0) \cap (\bar{\mathbb{C}} - \overline{W}_0) \neq \emptyset$. Based on the symmetry of $f_\lambda^{-1}(0)$ and $\bar{\mathbb{C}} - \overline{W}_0$, we have $f_\lambda^{-1}(0) \subset \bar{\mathbb{C}} - \overline{W}_0$. This will contradict the fact that $f_\lambda^{-1}(0) = H_\lambda(f_\lambda^{-1}(0)) \subset H_\lambda(\bar{\mathbb{C}} - \overline{W}_0) \subset W_0$.

Because no point on ∂W_0 can be mapped into W_0, we have $f_\lambda^{-1}(W_0) \subset W_0$ and $f_\lambda^{-1}(\overline{W}_0) \subset \overline{W}_0$. Take a point $z \in \partial W_0$; we have $\partial B_\lambda \subset J(f_\lambda) = \overline{\bigcup_{k \geqslant 0} f_\lambda^{-k}(z)} \subset \overline{W}_0$ and $\partial B_\lambda \subset \partial W_0$. Therefore, $\partial W_0 = \partial B_\lambda$.

Now, we show that ∂B_λ is a Jordan curve. If two different external rays, say $R_\lambda(t_1)$ and $R_\lambda(t_2)$, land at the same point $p \in \partial B_\lambda$, then $\overline{R_\lambda(t_1) \cup R_\lambda(t_2)}$ decomposes ∂B_λ into two parts. It turns out that $\partial W_0 \neq \partial B_\lambda$, which is a contradiction. □

The aim of this section is to prove Theorem 1.3, as follows:

Proof of Theorem 1.3. By Theorem 1.1 and Proposition 6.1, it suffices to show that f_λ satisfies the **BD** condition on ∂B_λ. First, we deal with three special cases:

Case 1. The critical orbit escapes to infinity.

Case 2. The parameter $\lambda \in \mathbb{R}^+$ and ∂B_λ contains no parabolic point.

Case 3. The map f_λ is critically finite.

In Case 1, $P(f_\lambda) \cap \partial B_\lambda = \emptyset$. Based on Proposition 6.2, f_λ satisfies the **BD** condition on ∂B_λ. For Case 2, by Proposition 7.3, either $P(f_\lambda) \cap \partial B_\lambda = \emptyset$ or $P(f_\lambda) \cap \partial B_\lambda = \{\beta_c\}$ or $P(f_\lambda) \cap \partial B_\lambda = \{\pm\beta_c\}$. In either case, β_c is a repelling fixed point of f_λ. By Proposition 6.1, f_λ satisfies the **BD** condition on ∂B_λ. For Case 3, f_λ satisfies the **BD** condition on ∂B_λ by Corollary 6.2.

In the remaining cases, we can use the Yoccoz puzzle to study the higher regularity of ∂B_λ. There are two remaining cases:

Case 4. ∂B_λ contains no critical point.

Case 5. $C_\lambda \subset \partial B_\lambda$ and all critical points in C_λ are non-recurrent.

In either case, by Proposition 4.1, we can find an admissible graph $\mathbf{G}_\lambda(\theta_1, \ldots, \theta_N)$. With respect to the Yoccoz puzzle induced by this graph, we consider the critical tableaux. For Case 4, there are two possibilities:

Case 4.1. Some $T(c)$ with $c \in C_\lambda$ is periodic.

Case 4.2. No $T(c)$ with $c \in C_\lambda$ is periodic.

For Case 4.1, we conclude from Proposition 7.2 that $\#(P(f_\lambda) \cap \partial B_\lambda) < \infty$. Because ∂B_λ contains no parabolic point, all periodic points in $P(f_\lambda) \cap \partial B_\lambda$ are repelling. Thus, based on Proposition 6.2, f_λ satisfies the **BD** condition on ∂B_λ.

For Case 4.2, we have already shown that $\mathrm{End}(c) = \bigcap_{d \geqslant 0} \overline{P_d(c)} = \{c\}$ for $c \in C_\lambda$ in the proof of Proposition 7.1. Thus, we can choose a d_0 large enough such that

$$\mathrm{Eucl.diam}\big(P_{d_0}(c)\big) < \mathrm{Eucl.dist}(c, \partial B_\lambda).$$

For $d \geqslant d_0$, let U_d be the union of all puzzle pieces of depth d that intersect with ∂B_λ and V_d be the interior of $\overline{U_d}$. For every $u \in \partial B_\lambda$, there is a number $\varepsilon_u > 0$ such that $B(u, \varepsilon_u) \subset V_{d_0}$. For any $m \geqslant 0$ and any component $U_m(u)$ of $f_\lambda^{-m}(B(u, \varepsilon_u))$ intersecting with ∂B_λ, $U_m(u) \subset V_{d_0+m} \subset V_{d_0}$. By the choice of d_0, the sequence $U_m(u) \to \cdots \to f_\lambda^{m-1}(U_m(u)) \to B(u, \varepsilon_u)$ meets no critical point of f_λ; thus, $f_\lambda^m : U_m(u) \to B(u, \varepsilon_u)$ is a conformal map. Therefore, in this case, f_λ satisfies the **BD** condition on ∂B_λ.

In the following, we deal with Case 5. Again, based on Proposition 7.1, $\mathrm{End}(c) = \{c\}$ for $c \in C_\lambda$. Thus, in this case one can verify that ∂B_λ contains no recurrent critical point if and only if all tableaux $T(c)$ with $c \in C_\lambda$ are non-critical. Based on Lemma 5.1, f_λ is critically finite. It follows from Corollary 6.1 that f_λ satisfies the **BD** condition on ∂B_λ. □

7.5. Corollaries

In this section, we present some corollaries of Theorem 1.1.

Proposition 7.5. *If ∂B_λ contains a parabolic cycle, then the multiplier of the cycle is* 1 *and the Julia set $J(f_\lambda)$ contains a quasi-conformal copy of the quadratic Julia set of $z \mapsto z^2 + 1/4$.*

Proof. Suppose $\mathcal{C} = \{z_0, f_\lambda(z_0), \ldots, f_\lambda^q(z_0) = z_0\}$ is a parabolic cycle on ∂B_λ. We will first consider the case $\lambda \in \mathbb{R}^+$, then deal with the case $\lambda \in \mathcal{H}$.

First, suppose $\lambda \in \mathbb{R}^+$. By Lemma 7.2 and Proposition 7.3, f_λ is 1-renormalizable at c_0 and $P(f_\lambda) \cap \partial B_\lambda \subset (-K_{c_0} \cup K_{c_0}) \cap \partial B_\lambda = \{\pm\beta_{c_0}\}$. Because a parabolic point must attract a critical point, we conclude that β_{c_0} is a parabolic fixed point of f_λ. Therefore, (f_λ, U, V) is quasi-conformally conjugate to a quadratic polynomial $z \mapsto z^2 + \mu$ with a β-fixed point that is also a parabolic point, thus $\mu = 1/4$. The conclusion follows in this case.

In the following, we deal with the case $\lambda \in \mathcal{H}$. Based on Proposition 4.1, we can find an admissible graph $\mathbf{G}_\lambda(\theta_1, \ldots, \theta_N)$. Based on Proposition 3.4, the parabolic cycle $\mathcal{C}$ avoids the graph $\mathbf{G}_\lambda(\theta_1, \ldots, \theta_N)$. With respect to the Yoccoz puzzle induced by this graph and with an argument similar to that used to prove Corollary 5.1, we conclude that there is a critical point $c \in C_\lambda$ and a point $z \in \mathcal{C}$ such that $P_d(z) = P_d(c)$ for all $d \geqslant 0$. Thus, the tableau $T(c)$ is periodic. Suppose the period of $T(c)$ is k. It is obvious that k is a divisor of q. By Lemma 5.1, when d_0 is large enough, the triple $(\epsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is either a k-renormalization of f_λ at c (in this case, $(\epsilon, p) = (1, k)$) or a $k/2$-$*$-renormalization of f_λ at c (in this case, $(\epsilon, p) = (-1, k/2)$). Moreover, the small filled Julia set $K_c = \mathrm{End}(c) = \bigcap_{d \geqslant 0} \overline{P_d(c)}$ and $z \in K_c \cap \partial B_\lambda$.

On the other hand, based on Lemma 5.3, there is a unique external ray $R_\lambda(t)$ landing at β_c, which is the β-fixed point of the renormalization $(\epsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$. Note that we have already proved that ∂B_λ is a Jordan curve; the intersection $\partial B_\lambda \cap \overline{P_d(c)}$ shrinks to a single point as $d \to \infty$. Thus, we have $K_c \cap \partial B_\lambda = \{\beta_c\}$. By the previous argument, $\beta_c = z$.

Based on the straightening theorem of Douady and Hubbard, $(\epsilon f_\lambda^p, P_{d_0+p}(c), P_{d_0}(c))$ is quasi-conformally conjugate to a quadratic polynomial $p_\mu(z) = z^2 + \mu$ in a neighborhood of the small filled Julia set K_c. For this quadratic polynomial, the β-fixed point is also a parabolic point, thus $\mu = 1/4$. Therefore, the Julia set $J(f_\lambda)$ contains a quasi-conformal copy of the quadratic

Julia set of $z \mapsto z^2 + 1/4$. Because the multiplier of the parabolic point of $z \mapsto z^2 + 1/4$ is 1, it turns out that $(\epsilon f_\lambda^p)'(z) = 1$, $(f_\lambda^k)'(z) = 1$ and $(f_\lambda^q)'(z) = 1$. $\square$

Proposition 7.6. *Suppose f_λ has no Siegal disk and the Julia set $J(f_\lambda)$ is connected, then every Fatou component is a Jordan domain.*

Proof. By Proposition 7.4 and the fact that $H_\lambda(B_\lambda) = T_\lambda$, we conclude that both T_λ and B_λ are Jordan domains.

If the critical orbit tends to ∞, then the Julia set is a Sierpinski curve that is locally connected, and all Fatou components are quasi-disks (by Proposition 6.1).

If the critical orbit remains bounded, then for any $U \in \mathcal{P} \setminus \{T_\lambda, B_\lambda\}$, there is a smallest integer $k \geqslant 1$ such that $f_\lambda^k : U \to T_\lambda$ is a conformal map. Thus, if two radial rays $R_U(\theta_1)$ and $R_U(\theta_2)$ land at the same point, then $R_{T_\lambda}(\theta_1) = f_\lambda^k(R_U(\theta_1))$ and $R_{T_\lambda}(\theta_2) = f_\lambda^k(R_U(\theta_2))$ also land at the same point. This implies that U is also a Jordan domain. If there are other Fatou components, then they are eventually mapped to a parabolic basin or an attracting basin. By Proposition 5.1, the map is either renormalizable or $*$-renormalizable. It is known that every bounded Fatou component of a quadratic polynomial without a Siegal disk is a Jordan disk; it turns out that all Fatou components of f_λ are Jordan disks in this case. $\square$

Proposition 7.7. *If f_λ has a Cremer point, then the Cremer point cannot lie on the boundary of any Fatou component. In other words, all Cremer points are buried on the Julia set.*

Proof. Suppose f_λ has a Cremer point z, then the Fatou set $F(f_\lambda) = \bigcup_{k \geqslant 0} f_\lambda^{-k}(B_\lambda)$. If z lies on the boundary of some Fatou component, then after iterations, one sees that $z \in \partial B_\lambda$. By Theorem 1, there is a periodic external ray $R_\lambda(t)$ landing at z. But this is a contradiction because, by the Snail Lemma, every periodic external ray can only land at a parabolic point or a repelling point (see [20]). $\square$

8. Local connectivity of the Julia set $J(f_\lambda)$

In this section, we study the local connectivity of the Julia set $J(f_\lambda)$. We will prove Theorem 1.3.

The proof is based on the 'Characterization of Local Connectivity' (Proposition 8.1 (see [31])) and the 'Shrinking Lemma' (Proposition 8.2 (see [29] or [17])), as follows.

Proposition 8.1. *A connected and compact set $X \subset \bar{\mathbb{C}}$ is locally connected if and only if it satisfies the following conditions:*

1. *Every component of $\bar{\mathbb{C}} \setminus X$ is locally connected.*
2. *For any $\epsilon > 0$, there are only a finite number of components of $\bar{\mathbb{C}} \setminus X$ with spherical diameter greater than ϵ.*

Proposition 8.2. *Let $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map and D be a topological disk whose closure $\overline{D}$ has no intersection with the post-critical set $P(f)$. Then, either $\overline{D}$ is contained in a Siegel disk or a Herman ring or for any $\epsilon > 0$ there are at most finitely many iterated preimages of D with spherical diameter greater than ϵ.*

Proof of Theorem 1.3. 1. If f_λ is geometrically finite, then $J(f_\lambda)$ is locally connected (see [29]). Otherwise, the Fatou set $F(f_\lambda) = \bigcup_{k \geqslant 0} f_\lambda^{-k}(B_\lambda)$. Because $\overline{B}_\lambda \cap P(f_\lambda) = \emptyset$, we conclude base on Shrinking Lemma that for any $\epsilon > 0$, there are at most finitely many iterated preimages of B_λ with spherical diameter greater than ϵ. Based on Proposition 8.1, $J(f_\lambda)$ is locally connected.

2. If f_λ is neither renormalizable nor $*$-renormalizable, then the parameter $\lambda \in \mathcal{H}$ by Lemma 7.2. We can assume that f_λ is not critically finite; otherwise, the Julia set is locally connected. Thus, based on Proposition 4.1, we can find an admissible graph. By Lemma 5.1, none of the tableaux $T(c)$ with $c \in C_\lambda$ are periodic. The local connectivity of $J(f_\lambda)$ follows from Proposition 7.1.

3. (The notations here are the same as in Section 7.3.) We need only consider the case when f_λ is not geometrically finite. In this case, the Fatou set $F(f_\lambda) = \bigcup_{k \geqslant 0} f_\lambda^{-k}(B_\lambda)$. Note that for any $z > 0$, $f_\lambda(z) \geqslant 2\sqrt{z^n \cdot \frac{\lambda}{z^n}} = 2\sqrt{\lambda} = v_\lambda^+$. Thus, $\{f_\lambda^k(v_\lambda^+);\ k \geqslant 0\} \subset [v_\lambda^+, \beta_{c_0}]$.

If $v_\lambda^+ = \beta'_{c_0}$, one can easily verify that the triple (f_λ, U, V) is quasi-conformally conjugate to the quadratic polynomial $z \mapsto z^2 - 2$, which is critically finite. Therefore, f_λ is also critically finite, and the Julia set is locally connected.

If $v_\lambda^+ > \beta'_{c_0}$, then $\overline{T}_\lambda \cap [v_\lambda^+, \beta_{c_0}] = \emptyset$ by Remark 7.1. Because $P(f_\lambda) \subset [-\beta_{c_0}, v_\lambda^-] \cup [v_\lambda^+, \beta_{c_0}] \cup \{\infty\}$, we have $\overline{T}_\lambda \cap P(f_\lambda) = \emptyset$. Based on Proposition 8.2, for any $\epsilon > 0$ there are at most finitely many iterated preimages of T_λ with spherical diameter greater than ϵ. Based on Proposition 8.1, the Julia set is locally connected. $\square$

Acknowledgments

The authors would like to thank the referees for their careful reading of the manuscript and their helpful comments. This research was supported by the National Natural Science Foundation of China (Grants Nos. 10831004, 10871047) and by the Science and Technology Commission of Shanghai Municipality (NSF Grant No. 10ZR1403700).

References

[1] L.V. Alhfors, Lectures on Quasiconformal Mappings, Univ. Lecture Ser., vol. 38, Amer. Math. Soc., 2006.
[2] B. Branner, J.H. Hubbard, The iteration of cubic polynomials, Acta Math. 169 (1992) 229–325.
[3] L. Carleson, T. Gamelin, Complex Dynamics, Springer-Verlag, New York, 1993.
[4] L. Carleson, P. Jones, J.-C. Yoccoz, Julia and John, Bull. Braz. Math. Soc. (N.S.) 25 (1994) 1–30.
[5] R. Devaney, Cantor webs in the parameter and dynamical planes of rational maps, Fields Inst. Commun. 53 (2008) 105–123.
[6] R. Devaney, Intertwined internal rays in Julia sets of rational maps, Fund. Math. 206 (2009) 139–159.
[7] R. Devaney, L. Keen, Complex Dynamics: Twenty-Five Years After the Appearance of the Mandelbrot Set, in: Contemp. Math., vol. 396, American Mathematical Society, 2006.
[8] R. Devaney, D. Look, D. Uminsky, The escape trichotomy for singularly perturbed rational maps, Indiana Univ. Math. J. 54 (2005) 1621–1634.
[9] A. Douady, Disques de Siegel et anneaux de Herman, in: Séminaire Bourbaki, vol. 1986/87, Astérisque 152–153 (1987) 4, Astérisque 151–172 (1988).
[10] A. Douady, J.H. Hubbard, Etude dynamique des polynômes complexes, Publications Mathématiques d'Orsay, 1984.
[11] A. Douady, J.H. Hubbard, On the dynamics of polynomial-like mappings, Ann. Sci. École Norm. Sup. 18 (1985) 287–343.
[12] L. Geyer, Linearizability of saturated polynomials, preprint, 2004.
[13] J. Graczyk, G. Swiatek, The Real Fatou Conjecture, Ann. of Math. Stud., vol. 144, Princeton Univ. Press, Princeton, NJ, 1998.

[14] J.H. Hubbard, Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz, in: L.R. Goldberg, A.V. Phillips (Eds.), Topological Methods in Modern Mathematics, Publish or Perish, Houston, TX, 1993, pp. 467–511.
[15] J. Kiwi, Real laminations and the topological dynamics of complex polynomials, Adv. Math. 184 (2004) 207–267.
[16] M. Lyubich, Dynamics of quadratic polynomials, I–II, Acta Math. 178 (1997) 185–297.
[17] M. Lyubich, Y. Minsky, Laminations in holomorphic dynamics, J. Differential Geom. 47 (1997) 17–94.
[18] C. McMullen, Automorphisms of rational maps, in: Holomorphic Functions and Moduli I, Springer-Verlag, 1988, pp. 31–60.
[19] C. McMullen, Complex Dynamics and Renormalization, Ann. of Math. Stud., vol. 135, Princeton Univ. Press, Princeton, NJ, 1994.
[20] J. Milnor, Dynamics in One Complex Variable, 2nd edition, Vieweg, 1999, 2000.
[21] J. Milnor, Local connectivity of Julia sets: Expository lectures in the Mandelbrot set, theme and variations, in: Tan Lei (Ed.), London Math. Soc. Lecture Note Ser., vol. 274, Cambridge Univ. Press, Cambridge, 2000, pp. 67–116.
[22] W. Peng, W. Qiu, P. Roesch, L. Tan, Y. Yin, A tableau approach of the KSS nest, Conform. Geom. Dyn. 14 (2010) 35–67.
[23] C.L. Petersen, Local connectivity of some Julia sets containing a circle with an irrational rotation, Acta Math. 177 (1996) 163–224.
[24] K. Pilgrim, L. Tan, Rational maps with disconnected Julia set, Astérisque 261 (2000) 349–384.
[25] W. Qiu, Y. Yin, Proof of the Branner–Hubbard conjecture on cantor Julia sets, Sci. China Ser. A 52 (2009) 45–65.
[26] P. Roesch, On local connectivity for the Julia set of rational maps: Newton's famous example, Math. Ann. 168 (2008) 1–48.
[27] P. Roesch, Y. Yin, Bounded critical Fatou components are Jordan domains for polynomials, arXiv:0909.4598v2.
[28] M. Shishikura, On the quasiconformal surgery of rational functions, Ann. Sci. École Norm. Sup. 20 (1987) 1–29.
[29] L. Tan, Y. Yin, Local connectivity of the Julia set for geometrically finite rational maps, Sci. China Ser. A 39 (1996) 39–47.
[30] X. Wang, Dynamics of McMullen maps and Thurston-type theorems for rational maps with rotation domains, Thesis, Fudan University and Université d'Angers, 2011.
[31] G. Whyburn, Analytic Topology, Amer. Math. Soc. Colloq. Publ., vol. 28, 1942.
[32] Y. Yin, Y. Zhai, No invariant line fields on cantor Julia sets, Forum Math. 22 (2010) 75–94.
[33] J.-C. Yoccoz, Petits diviseurs en dimension 1, Astérisque 231 (1995) 3–88.